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Linear Ordinary Self-Adjoint Differential Equations of the Second Order.

By Anna Pell Wheeler.

In 1914 Lichtenstein * made connection, without the intermediary of the theory of integral equations, between the theory of linear differential systems of the second order and the theory of linear equations in infinitely many unknowns. Although there are many points of contact between the method used by Lichtenstein and the one used in this paper, the methods are not identical. The method employed here can be applied with only slight modifications to other systems, for example, singular differential systems,† and partial differential systems, elliptic or hyperbolic. The method can be extended to yield more general results both for the orthogonal and polar cases of ordinary differential systems.

I. *Existence of Characteristic Numbers.*

1. *Orthogonal Case.* $\nu_m \neq 0$. The differential system to be considered is

$$(1)\qquad \begin{aligned} u''(x) + q(x)u(x) + \lambda k(x)u(x) &= 0, \\ \alpha u(0) + \alpha' u'(0) &= 0, \\ \beta u(\pi) + \beta' u'(\pi) &= 0, \end{aligned}$$

where the function $q(x)$ is continuous, the function $k(x)$ is continuous and positive in the interval $(0, \pi)$, and α, α', β, β' are constants such that

$$|\alpha| + |\alpha'| \neq 0, \quad |\beta| + |\beta'| \neq 0.$$

The characteristic functions $\phi_m(x)$ corresponding to the characteristic numbers ν_m^2 of the system

* L. Lichtenstein, "Zur Analysis der unendlichvielen Variabeln. I. Entwicklungssätze der Theorie gewöhnlicher linearer Differentialgleichungen zweiter Ordnung," *Rendiconti del Circolo Matematico di Palermo*, Vol. 38 (1914), pp. 113-166. See also J. Tamarkine, "Problème du développement d'une fonction arbitraire en séries de Sturm-Liouville," *Comptes Rendus*, Vol. 156 (1913), pp. 1589-91.

† Marion Cameron Gray, "A Boundary Value Problem of Ordinary Self-Adjoint Differential Equations with Singularities." To appear in a later number of the *American Journal of Mathematics*.

$$
\begin{aligned}
&u''(x) + \nu_m^2 u(x) = 0,\\
&\alpha u(0) + \alpha' u'(0) = 0,\\
&\beta u(\pi) + \beta' u'(\pi) = 0,
\end{aligned}
\tag{2}
$$

form an orthogonal, normed and closed system of functions

$$
\begin{aligned}
&\phi_m(x) = c_1 \cos \nu_m x + c_2 \sin \nu_m x,\\
&\alpha c_1 + \alpha' \nu_m c_2 = 0,\\
&(\beta \cos \nu_m \pi - \beta' \nu_m \sin \nu_m \pi) c_1 + (\beta \sin \nu_m \pi + \beta' \nu_m \cos \nu_m \pi) = 0,
\end{aligned}
$$

where * $\nu_m = m + \gamma_m$, $\gamma_m \to 0$ if α' and β' are both $\neq 0$, $\nu_m = m + \frac{1}{2} + \gamma_m$, if one, but not both, of α' and $\beta' = 0$, and $\nu_m = 1 + m + \gamma_m$ if $\alpha' = \beta' = 0$. The orthogonal system affords a means of passing from a function space F of functions which have continuous derivatives of the second order and satisfy the boundary conditions of (1), to a space H_1 in the Hilbert space H of infinitely many variables, and also to pass back. The known results for linear equations in the space H thus yield results for equations (1) in the space F.

Multiply the differential equation of (1) by $\phi_m(x)$ and integrate between 0 and π

$$
\int u''\phi_m + \int qu\phi_m + \lambda \int ku\phi_m = 0 \qquad (m = 1, 2, \cdots).
$$

Integrating twice the first integral by parts, and expanding the other two integrals by means of the system $\{\phi_m(x)\}$ we obtain

$$
-\nu_m^2 \int u\phi_m + \sum_{n=1}^{\infty} q_{mn} \int u\phi_n + \lambda \sum_{n=1}^{\infty} k_{mn} \int u\phi_n = 0 \quad (m = 1, 2, \cdots) \tag{3}
$$

where the matrices $(q_{mn} = \int \phi_m q \phi_n)$ and $(k_{mn} = \int \phi_m k \phi_n)$ are limited and symmetric.

In this section we assume that no ν_m is equal to zero. The series $\sum_{m,n} (q^2_{mn}/\nu_m^4)$ and $\sum_{m,n} (k_{mn}/\nu_m^4)$ are convergent, and the matrices (q_{mn}/ν_m^2) and (k_{mn}/ν_m^2) are completely continuous. Therefore if the differential system (1) has a continuous solution $u(x)$, the sequence $x_m = \int u\phi_m$ of finite norm satisfies the system of linear equations

$$
x_m - \sum_n (q_{mn}/\nu_m^2) x_n - \lambda \sum_n (k_{mn}/\nu_m^2) x_n = 0 \qquad (m = 1, 2, \cdots), \tag{4}
$$

in which the matrices involved are completely continuous. The system (4) can be put into a form for which results are known.

---

* M. Bôcher, "Boundary Problems in One Dimension," *International Congress of Mathematicians*, Cambridge, 1912, p. 178.

Consider first the case in which the system

$$x_m - \sum_n (q_{mn}/\nu_m^2)x_n = 0 \qquad (m = 1, 2, \cdots) \tag{5}$$

has no solutions of finite norm. The matrix $(e_{mn} - (q_{mn}/\nu_m^2))$, where $e_{mn} = 0$ if $m \neq n$ and $= 1$ if $m = n$, has a limited reciprocal (r_{mn}), that is

$$\sum_p r_{mp}\,[e_{pn} - (q_{pn}/\nu_p^2)] = e_{mn}, \tag{6}$$

and the system (4) is equivalent to the system

$$x_m - \lambda \sum_n [\sum_p (r_{mp}k_{pn}/\nu_p^2]x_n = 0 \qquad (m = 1, 2, \cdots). \tag{7}$$

The matrix (r_{mn}/ν_m^2), which is completely continuous, is symmetric,* for it is the reciprocal of the symmetric matrix $(e_{mn}\nu_m^2 - q_{mn})$. The function $k(x)$ by hypothesis is positive in the interval $(0, \pi)$. It is easy to show that the matrix $(k_{mn}/\nu_m\nu_n)$ is positive definite, and then it can be argued that the matrix (k_{mn}) itself is. Hence † there exist characteristic numbers λ_a, which are necessarily real, and corresponding solutions l_{am} of finite norm of

$$l_{am} - \lambda_a \sum_n (\sum_p r_{mp}k_{pn}/\nu_p^2)\; l_{an} = 0 \qquad (m = 1, 2, \cdots). \tag{8}$$

The system of equations with the transposed matrix has the solutions $\sum_p k_{mp}l_{ap} = m_{am}$ corresponding to λ_a, for

$$\sum_p k_{mp}l_{ap} - \lambda_a \sum_{p,q} (k_{mp}r_{qp}/\nu_p^2)\;(\sum_n k_{qn}l_{an}) = 0 \qquad (m = 1, 2, \cdots)$$

is obtained by multiplying the system (8) by k_{np} and summing, and using the symmetry of the matrix (r_{mn}/ν_m^2). The solutions l_{am} and m_{am} form a biorthogonal system

$$\sum_m l_{am}m_{\beta m} = e_{a\beta}, \tag{9}$$

and further

$$\sum_{p,m,n} (r_{mp}k_{pn}/\nu_p^2)x_n y_m = \sum_a \;(\sum_m l_{am}y_m \sum_n m_{an}x_n)/\lambda_a \tag{10}$$

for every $\{x_m\}$ of finite norm, and every $\{y_m = \Sigma_n k_{mn}x_n\}$, and since the matrix (k_{mn}) has a limited reciprocal the relation (10) holds for every $\{x_m\}$ and $\{y_n\}$ of finite norm.

---

* If $\Sigma_p a_{mp}b_{pn} = e_{mn}$, then $\Sigma_p b_{pm}(\Sigma_q a_{pq}b_{qn}) = b_{mn}$, and if the matrix (a_{mn}) is symmetric and the order of summation may be interchanged, the matrix (b_{mn}) is symmetric.

† A. J. Pell, "Linear Equations with Unsymmetric Coefficients," *Transactions of the American Mathematical Society*, Vol. 20 (1919), pp. 23-39.

The characteristic numbers can have a limiting point only at infinity. The system (8) is equivalent to

$$(11) \qquad -\nu_m^2 l_{\alpha m} + \sum_n q_{mn} l_{\alpha n} + \lambda_\alpha \sum_n k_{mn} l_{\alpha n} = 0 \qquad (m = 1, 2, \cdots).$$

The matrix $(m_{\alpha m} = \Sigma_p k_{mp} l_{\alpha p})$ is limited,* and hence the matrix $(l_{\alpha m})$ is also, since the matrix (k_{mn}) has a limited reciprocal. The equations (11) show that the sequence $\{\lambda_\alpha \Sigma_n k_{mn} l_{\alpha n}\}$ is of finite norm for every m and that $\{\nu_m^2 l_{\alpha m}\}$ is of finite norm for every α.

The function $u_\alpha(x)$ defined by the absolutely uniformly convergent series

$$u_\alpha(x) = \sum_m l_{\alpha m} \phi_m(x) = \sum_m \nu_m^2 l_{\alpha m} \, (\phi_m(x)/\nu_m^2),$$

has a continuous derivative, given by the absolutely uniformly convergent series

$$\sum_m l_{\alpha m} \phi'(x) = \sum_m \nu_m^2 l_{\alpha m} \, (\phi'_m(x)/\nu_m^2).$$

The functions $u_\alpha(x)$ and $u_\alpha'(x)$ satisfy the boundary conditions of (1). The function $u_\alpha(x)$ has a continuous second derivative, for, let $f = (q + \lambda_\alpha k) u_\alpha$, then

$$\int f\phi_m = \int \left(\int_0^x \int_0^{x_1} f(x_2)\, dx_2 dx_1 + c_1 x + c_2 \right) \phi_m''(x)\, dx$$

$$= -\nu_m^2 \int \left(\int_0^x \int_0^{x_1} f(x_2)\, dx_2 dx_1 + c_1 x + c_2 \right) \phi_m(x)\, dx = \nu_m^2 \int u_\alpha \phi_m,$$

and

$$-u_\alpha(x) = \int_0^x \int_0^{x_1} f(x_2)\, dx_2 dx_1 + c_1 x + c_2.$$

Moreover, the function $u_\alpha(x)$ satisfies the system (1) for the characteristic number λ_α.

Consider next the case in which there exists a sequence $\{z_m\}$ of finite norm which satisfies (5). Since the matrix in (5) is completely continuous there can be only a finite number of such solutions. Let us assume there is only one; the case in which there is more than one can be carried through in a similar manner. There exists a limited matrix (r_{mn}) such that

$$\sum_p r_{mp} \, [e_{pn} - q_{pn}/\nu_p^2)] = e_{mn} - z_m z_n,$$

where $\Sigma_n z_n^2 = 1$. As before, the matrix (r_{mn}/ν_n^2) is symmetric closed and completely continuous. The system to be considered now is

---

* A. J. Pell, *l. c.*, pp. 30-31.

$$(12)\qquad x_m - z_m \sum_n z_n x_n + \lambda \sum_n \sum_p (r_{mp}k_{pn}/\nu_p^2)x_n = 0 \qquad (m = 1, 2, \cdots),$$
$$\sum_{m,n} z_m k_{mn} x_n = 0.$$

There exist * characteristic numbers λ_α and corresponding characteristic solutions $l_{\alpha m}$ for this system. To the value $\lambda_\alpha = 0$, there corresponds the solution $z_m = l_{\alpha m}$. The remainder follows as in the preceding case, and the existence of characteristic numbers for the differential system (1), under conditions specified, has been established by means of the properties of the solutions of the system (11). The system of functions $\{u_\alpha(x)\}$ has the orthogonal property, or the systems $\{u_\alpha(x)\}$ and $\{k(x)u_\alpha(x)\}$ have the biorthogonal property

$$(13)\qquad \int k\, u_\alpha u_\beta = e_{\alpha\beta}.$$

2. *Polar Case.* $\nu_m \neq 0$. Let the function $k(x)$ change sign in the interval $(0, \pi)$. The matrix (k_{mn}) is then not positive definite, and in order to conclude the existence of real characteristic numbers for the equations (4) or (7) we assume that the matrix (r_{mn}/ν_n^2) satisfying (6) exists, that it is positive definite, and for the present that there exists no sequence $\{f_n\}$ of finite norm such that $\sum (r_{mn}/\nu_n^2) f_n = 0$. A sufficient condition for this is that the matrix $(e_{mn}\nu_n^2 - q_{mn})$ be positive definite for any sequence $\{x_m\}$ such that $\{\nu_m x_m\}$ is of finite norm. This condition is certainly fulfilled when $q(x) < 0$, and also when the matrix $(e_{mn} - q_{mn}/\nu_m\nu_n)$ is positive definite. The latter is true if the positive † characteristic numbers of $(q_{mn}/\nu_m\nu_n)$ are < 1.

The matrix (r_{mn}/ν_n^2) is limited, symmetric and positive definite, hence there exist real characteristic numbers λ_α and corresponding solutions $m_{\alpha m}$ of the system

$$(14)\qquad m_{\alpha m} - \lambda_\alpha \sum_{n,p} (k_{mp}r_{np}/\nu_p^2) m_{\alpha n} = 0 \qquad (m = 1, 2, \cdots),$$

and solutions $l_{\alpha m}$ of finite norm of the equations (8), where

$$(15)\qquad l_{\alpha m} = \sum_n (r_{mn}/\nu_n^2) m_{\alpha n} = \sum_n (r_{nm}/\nu_m^2) m_{\alpha n},$$

$$(15')\qquad m_{\alpha m} = \sum_n (\nu_n^2 e_{mn} - q_{mn}) l_{\alpha m} = \lambda_\alpha \sum_n k_{mn} l_{\alpha n}.$$

The two systems $\{l_{\alpha m}\}$ and $\{m_{\alpha m}\}$ form a biorthogonal system

$$\sum_n l_{\alpha m} m_{\beta m} = e_{\alpha\beta},$$

---

* A. J. Pell, "Linear Equations with Two Parameters," *Transactions of the American Mathematical Society*, Vol. 23 (1922), pp. 201-203.

† L. Lichtenstein, *l. c.*, p. 129.

and the expansion (10) holds for every $\{y_m\}$ of finite norm and every $\{x_m\}$ such that $\{\Sigma_m m_{am}x_m\}$ is of finite norm, in particular, if $x_m = \Sigma_n(r_{mn}/v_n^2)z_n$. On account of (15) the matrix (l_{am}) is limited. The characteristic numbers can have no finite limiting point. The sequence $\{v_m^2 l_{am}\}$ is of finite norm for every α, and the sequence $\{\lambda_a m_{am}\}$ is of finite norm for every m. As before we obtain the existence of characteristic numbers λ_a for the differential system (1), and corresponding characteristic functions $u_a(x)$ where

$$l_{am} = \int u_a \phi_m,$$

and

$$m_{am} = \int (u_a'' - q u_a)\phi_m = \lambda_a \int k u_a \phi_m.$$

The property corresponding to the orthogonal or biorthogonal property is

$$\lambda_a \int u_a k u_\beta = = e_{a\beta},$$

which is equivalent to

$$\int \bar{u}_a k \bar{u}_\beta = (|\lambda_a|/\lambda_a) e_{a\beta},$$

where $\bar{u}_a(x) = |\lambda_a|^{\frac{1}{2}} u_a(x)$.

Consider next the case in which there exists a sequence $\{z_m\}$ which satisfies (5). Again we obtain the system (12), in which the matrix (r_{mn}/v_n^2) may be assumed to be positive definite, and $z_m = \Sigma_n(r_{mn}/v_n^2)z_n$. A symmetrizing matrix is $(r^*_{mn}) = (\Sigma k_{mp}(r_{pq}/v^2_q)k_{qn})$, which is positive definite if $k(x)$ vanishes only at the points of a set of measure zero, and is orthogonal only to sequences $\{f_m\}$ for which $\Sigma_n k_{mn} f_n = 0$. Hence real characteristic numbers λ_a exist. The solutions l_{am} and

$$m_{am} = \Sigma_n r^*_{mn} l_{an} = \Sigma_p k_{mp}(l_{ap} - z_p \Sigma_q z_q l_{aq})$$

form a biorthogonal system. The sequence $\{\Sigma_n k_{mn} z_n\}$ is orthogonal to $\{l_{am}\}$, and if $\Sigma z_m k_{mn} z_n = 0$, $\{\Sigma_p r^*_{mp} z_p\}$ is also orthogonal to $\{l_{am}\}$. In place of (10) we have, for any $\{x_m\}$ and $\{y_n\}$ of finite norm, if $\Sigma z_m k_{mn} z_n \neq 0$,

$$(10') \qquad \sum_{m,p,q,n} x_m r^*_{mp}(r_{pq}/v_q^2) k_{qn} y_n - \frac{\sum_{m,n} x_m r^*_{mn} z_n \sum_{m,n} z_m r^*_{mn} y_n}{\sum_{m,n} z_m k_{mn} z_n} = \sum_a \frac{\sum_m m_{am} x_m \sum_n m_{an} y_n}{\lambda_a},$$

and if $\Sigma z_m k_{mn} z_n = 0$,

$$(10'') \quad \sum_{m,p,q,n} x_m r^*_{mp}(r_{pq}/\nu_q^2)k_{qn}y_n - \frac{\sum_{m,n} x_m r^*_{m,n}z_n \sum z_m r^*_{mn}Y_n + \sum z_m r^*_{mn}y_n \sum x_m r^*_{mn}Z_n}{\sum_{m,n} z_m r^*_{mn}z_n} = \sum_a \frac{\sum_m m_{am}x_m \sum_n m_{an}y_n}{\lambda_a},$$

where $P_n = \sum (r_{ns}/\nu_s^2)k_{sn}p_n$, and $m_{an} = \sum_n r^*_{mn}z_n$ corresponds to

$$\lambda_a = \frac{\sum_{m,n} z_m r^*_{mn}z_n}{\sum_{m,n} z_m r^*_{mn}Z_n}.$$

The existence of solutions of system (1) follows as before.

3. *Case in which* $\nu_1 = 0$. If $\nu_1 = 0$, the system of equations (3) takes the form

$$-\sum_{n=1}^{\infty} q_{1n}x_n = +\lambda \sum_{n=1}^{\infty} k_{1n}x_n,$$

$$\nu_m^2 x_m - \sum_{n=1}^{\infty} q_{mn}x_n = +\lambda \sum_{n=1}^{\infty} k_{mn}x_n \qquad (m = 2, 3, \cdots).$$

If $q(x) \not\equiv 0$ the matrix on the left has a completely continuous reciprocal (r_{mn}) given by

$$r_{mn} = r^*_{mn} + (1/a) \sum_{p=2}^{\infty} r^*_{mp}q_{p1} \sum_{p=2}^{\infty} r^*_{np}q_{p1} \qquad (m, n = 2, 3, \cdots),$$

$$r_{m1} = r_{1m} = -(1/a) \sum_{p=2}^{\infty} r^*_{mp}q_{p1} \qquad (m = 2, 3, \cdots),$$

$$r_{11} = 1/a,$$

where

$$a = -q_{11} + \sum_{m,n=2}^{\infty} q_{1m}r^*_{mn}q_{n1},$$

and (r^*_{mn}) is the reciprocal of $(\nu_m^2 e_{mn} - q_{mn})$, $(m, n = 2, 3, \cdots)$, assuming that $-\nu_m^2 z_m + \Sigma_n q_{mn}z_n = 0$ $(m = 2, 3, \cdots)$ has no solution of finite norm. The matrix above is symmetric and positive definite if (r^*_{mn}) is. The remainder follows as before for the orthogonal and for the polar case. And if the system $-\nu_m^2 z_m + \Sigma_n q_{mn}z_n = 0$ has solutions modifications can be made as in §§ 1 and 2.

If the function $q(x)$ is identically zero the equations (3) become

$$\lambda \sum_1^{\infty} k_{1n}x_n = 0,$$

$$-\nu_m^2 x_m + \lambda \sum_1^{\infty} k_{mn}x_n = 0 \qquad (m = 2, 3, \cdots).$$

It is obvious that $\lambda = 0$ is a characteristic number. For $k_{11} \neq 0$, the system above is equivalent to

$$-\nu_m^2 x_m + \lambda \sum_2^\infty [k_{mn} - (k_{m1}k_{n1}/k_{11})]x_n = 0 \quad (m = 2, 3, \cdots),$$

$$k_{11}x_1 = -\sum_2^\infty k_{1n}x_n.$$

This system is easily handled directly. The system of characteristic functions $u_\alpha(x)$ includes the function $u_1(x) = 1$, and the adjoint system of $u_\alpha(x)$ is $v_1 = k/\int k$, $v_\alpha = -\lambda_\alpha k u_\alpha$ $(\alpha \neq 1)$.

For $k_{11} = 0$, the system is equivalent to

$$-\nu_m^2 x_m + \lambda \sum_{n=2}^\infty k_{mn}x_n + \lambda k_{m1}x_1 = 0 \qquad (m = 2, 3, \cdots),$$

$$\sum_2^\infty k_{1n}x_n = 0,$$

or to

$$-\nu_m^2 x_m + \lambda \sum_{n=2}^\infty [k_{mn} - (1/a)\sum_{p=2}^\infty \frac{k_{1p}k_{pn}}{\nu_p^2}\, k_{m1} - (1/a)\sum_{p=2}^\infty \frac{k_{1p}k_{pm}}{\nu_p^2}\, k_{n1}]x_n = 0$$

$$(m = 2, 3, \cdots),$$

$$x_1 = -(1/a)\sum_{p,n=2}^\infty (k_{1p}k_{pn}/\nu_p^2)x_n,$$

$$a = \sum_{p=2}^\infty k_{1p}^2/\nu_p^2.$$

Again this system can be easily handled directly. The function k is orthogonal to all the characteristic functions including $u_1 = 1$. The adjoint system is $v_1 = 1/\pi$, $v_\alpha = -\lambda_\alpha k u_\alpha$ $(\alpha \neq 1)$.

II. *Expansion of Arbitrary Functions.*

4. *Orthogonal Case.* Since the matrix

$$\left(\sum_p (r_{mp}k_{pn}/\nu_p)\right)$$

is closed, there is an infinite number of characteristic numbers λ_α, and both the systems $\{l_{\alpha m}\}$ and $\{m_{\alpha m} = \Sigma_p k_{mp}l_{\alpha p}\}$ are closed, if the solutions of (5) are included. As a consequence it follows that

$$\sum_\alpha l_{\alpha m}m_{\alpha n} = e_{mn},$$

and this is equivalent to

$$\sum_\alpha \left(\sum_m l_{\alpha m}x_m\right)\left(\sum_n m_{\alpha n}y_n\right) = \sum x_m y_m$$

for every $\{x_m\}$ and $\{y_n\}$ of finite norm. Hence if $f(x)$ and $g(x)$ are continuous functions in $(0, \pi)$

$$\sum_a \int u_a g \int k f u_a = \int f g,$$

and if the series

$$\sum_a u_a(x) \int k f u_a$$

converges uniformly, it converges to $f(x)$. We proceed to obtain some sufficient conditions to ensure the uniform convergence of these series.

It follows from equations (11) that

$$-\nu_m^2 \int u_a \phi_m / \lambda_a + \int g u_a \phi_m / \lambda_a + \int k u_a \phi_m = 0, \qquad (m = 1, 2, \cdots),$$

and hence that the matrices $(\nu_m^2 \int u_a \phi_m / \lambda_a)$ and $(\nu_m^2 \int u_a \phi_m / |\lambda_a|)$ are limited, $\lambda_a = 0$ and $\nu_m = 0$ being excluded. The matrix $(\nu_m \int u_a \phi_m / |\lambda_a|^{1/2})$ is limited, for

$$\sum_a \frac{\nu_m \int u_a \phi_m}{|\lambda_a|^{1/2}} \quad \frac{\nu_n \int u_a \phi_n}{|\lambda_a|^{1/2}} = \nu_m \left(\sum_a \int u_a \phi_m \cdot \frac{\nu_m^2 \int u_a \phi_n}{|\lambda_a|} \right) \frac{1}{\nu_n},$$

are the elements of a symmetric matrix, equal to a matrix $(\nu_m a_{mn}/\nu_n)$ where (a_{mn}) is a limited matrix.* The same method of proof applies to show that the matrices

$$\left(\frac{|\nu_m|^t \int u_a \phi_m}{|\lambda_a|^{t/2}} \right), \quad \left(\frac{|\lambda_a|^{(2-t)/2} \int k u_a \phi_m}{|\nu_m|^{2-t}} \right)$$

are limited, where $t = 1/2^s$ and s is any positive integer. A repetition shows that the matrices are limited for $0 < t = 1/2^{s_1} + 1/2^{s_2} + \cdots + 1/2^{s_p} < 2$, where the exponents are all positive. If $f(x)$ is a continuous function

$$-\sum_m (\nu_m/|\lambda_a|^{1/2})^t \int u_a \phi_m \cdot \nu_m^{2-t} \int f \phi_m + \int g u_a f / |\lambda_a|^t + (|\lambda_a|/\lambda_a) |\lambda_a|^{(2-t)/2} \int k f u_a = 0,$$

and hence if $\{\nu_m^{2-t} \int f \phi_m\}$ is of finite norm, so also is $\{|\lambda_a|^{(2-t)/2} \int k f u_a\}$. If $\{\nu_m^r \int f \phi_m\}$, where $r < 2$, is of finite norm, $\{\nu_m^t \int f \phi_m\}$ is of finite norm and the norm is less than or equal to the norm of the first, where $t < r$ and has the form above. The sequences $\{|\lambda_a|^{t/2} \int k f u_a\}$ are of finite norm for these values of t and the norms are less than $M \int f^2$. The sequence $\{|\lambda_a|^{r/2} \int k f u_a\}$ is of finite norm if the sequence $\{\nu_m^r \int f \phi_m\}$ is of finite norm, and conversely.

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* A. J. Pell, "Linear Equations with Unsymmetric Systems of Coefficients," *Transactions of the American Mathematical Society*, Vol. 20 (1919), pp. 35-36.

From the equality

$$\frac{u_a(x)}{|\lambda_a|^{r/2}} = \sum_m \frac{\int u_a \phi_m}{|\lambda_a|^{r/2}} \cdot \nu_m{}^r \frac{\phi_m(x)}{\nu_m{}^r},$$

it is seen that $\{u_a(x)/|\lambda_a|^{r/2}\}$ is of finite norm if $\{\phi_m(x)/\nu_m{}^r\}$ is of finite norm. The converse is also true. Hence *any function* $f(x)$ *such that* $\{\nu_m{}^r \int f\phi_m\}$ *is of finite norm, where* $\{\phi_m(x)/\nu_m{}^r\}$ *is of finite norm and* $r < 2$, *can be expanded into an absolutely convergent series in terms of the characteristic functions* $u_a(x)$ *of the differential system* (1)

$$f(x) = \sum_a u_a(x) \int k f u_a.$$

A more intimate study of matrices such as $(\nu_m{}^r \int u_a \phi_m/|\lambda_a|^{r/2})$ would lead to more general expansion theorems.

5. *Polar Case.* If the equations (5) have no solutions, the relations (15) and (15′) may be given the form

$$\nu_m l_{am} = \sum_p (r_{pm}\nu_p/\nu_m)\, m_{ap}/\nu_p, \qquad m_{an}/\nu_n = \sum_p [e_{pn} - (q_{pn}/\nu_n\nu_p)]\, l_{ap}\nu_p.$$

Since both the matrices $(r_{mn}\nu_n/\nu_m)$ and $[e_{mn} - (q_{mn}/\nu_m\nu_n)]$ are symmetric, limited and positive definite, and since $\Sigma_m \nu_m l_{am} m_{\beta m}/\nu_m = e_{a\beta}$, the matrices $(\nu_m l_{am})$ and (m_{an}/ν_n) are limited. Hence the sequence of functions $\{u_a(x)\}$ which is given by

$$u_a(x) = \sum_m (\phi_m(x)/\nu_m) \cdot \nu_m l_{am},$$

is of finite norm, and the sequence $\{\lambda_a \int k g u_a = \Sigma_n m_{an} x_n\}$ is of finite norm for every function $g(x)$ such that $\{\nu_m \int g\phi_m\}$ is of finite norm. For such a function $g(x)$ the series

$$\sum_a u_a(x)\, \lambda_a \int k g u_a \tag{16}$$

is absolutely uniformly convergent. A consequence of (10) is that the system $\{l_{am}\}$ is closed, and the only sequence $\{f_m\}$ of finite norm, which is orthogonal to $\{m_{an}\}$ satisfies $\Sigma_n k_{mn} f_n = 0$. If we denote by $h(x)$ the function

$$h(x) = g(x) - \sum_a \phi_a(x)\, \lambda_a \int k g u_a,$$

the function $k(x)\, h(x)$ is orthogonal to the functions $u_a(x)$, and is therefore zero. Therefore *if* $g(x)$ *satisfies the conditions that* $\{\nu_m \int g\phi_m\}$ *be of finite norm*

$$k(x)\, g(x) = \sum_a k(x) u_a(x)\, \lambda_a \int k g u_a = \sum_a (|\lambda_a|/\lambda_a) \cdot k(x)\bar{u}_a(x) \int k g \bar{u}_a,$$

and the series converges absolutely uniformly.

If the equations (5) has a solution $\{z_m\}$ of finite norm the relations (15) and (15′) are to be replaced by

$$v_m(l_{am} - z_m \sum_n l_{an}z_n) = \lambda_a \sum_p (r_{pm}/v_m)v_p \cdot m_{ap}/v_p,$$

$$m_{an}/v_n = \sum_p [e_{np} - (q_{np}/v_n v_p)]\, l_{ap}v_p.$$

As above it follows that the matrices $v_m(l_{am} - z_m \Sigma_n l_{an}z_n)$ and (m_{an}/v_n) are limited. Since the sequences $\{v_m z_m\}$ and $\{\Sigma l_{an}z_n\}$ are of finite norm, the matrix $(v_m l_{am})$ is also limited. Again we get the absolutely uniform convergence of (16) under the same conditions as above. From (10′) it follows that if $\{h_m\}$ is such that $\Sigma h_m k_{mn} l_{an} = 0$, $\Sigma h_m k_{mn} z_n = 0$, then

$$\Sigma r^*_{mp} r_{pq}/v_q^2 k_{qn} h_n = c \Sigma r^*_{mn} z_n,$$

and hence

$$\Sigma k_{mp} r_{pq}/v_q^2 k_{qn} h_n = c \Sigma k_{mn} z_n,$$

From this,

$$\Sigma h_m k_{mp} r_{pq}/v_q^2 k_{qn} h_n = 0 \text{ and } \Sigma k_{mn} h_n = 0.$$

Therefore we have the result that *if* $\Sigma z_m k_{mn} z_n \neq 0$ *and the sequence* $\{v_m \int g\phi_m\}$ *is of finite norm*

$$k(x)\, g(x) = \sum_a k(x)\, u_a(x)\, \lambda_a \int gku_a + k(x)\, z(x) \int zkg/\int zkz,$$

and the series converges absolutely uniformly.

If $\Sigma z_m k_{mn} z_n = 0$, then $\Sigma h_m k_{mn} l_{an} = 0$, $\Sigma h_m k_{mn} z_n = 0$, $\Sigma h_m r^*_{mn} z_n = 0$ implies that $\Sigma k_{mn} h_n = 0$. *In this case, if the function* $g(x)$ *satisfies the same conditions as above*

$$k(x)\, g(x) = \sum_a k(x)\, u_a(x) \lambda_a \int gku_a + k(x)\, z(x) \int gk\psi/\int zk\psi + k(x)\, \psi(x) \int gkz/\int zk\psi,$$

$$k(x)z(x) = \psi''(x) + g(x)\psi(x),$$

$$\alpha\psi'(0) + \beta\psi(0) = 0,$$

$$\alpha'\psi'(\pi) + \beta'\psi(\pi) = 0,$$

and the series converges absolutely uniformly.

The equations arising in I § 3 were reduced to equations of the same general character as in §§ 1, 2, and the theory for the expansions need not be carried through.

If a more general differential equation

$$\frac{d}{dx}\left(p \frac{du}{dx} \right) + qu + \lambda k u = 0$$

were given, where $p(x)$ is positive in $(0, \pi)$, the work would be similar. In place of the matrix $(\nu_m^2 e_{mn})$ and its completely continuous reciprocal, there would be the matrix $(\int (p\phi_m')'\phi_n)$ with its completely continuous reciprocal.

BRYN MAWR COLLEGE.

The Singularities of a Function Defined by a Dirichlet Series.*

By D. V. Widder.†

1. *Introduction.* It is the purpose of the present paper to give a demonstration of a theorem enunciated by the author in a recent number of the *Comptes Rendus.*‡ The theorem is a generalization of a familiar theorem of J. Hadamard concerning the multiplication of the singularities of two functions defined by their power series developments. Thus, if the series

$$\sum_{n=0}^{\infty} a_n z^n, \qquad \sum_{n=0}^{\infty} b_n z^n \tag{1}$$

define functions with singularities at points a and b respectively, the function defined by

$$\sum_{n=0}^{\infty} a_n b_n z^n$$

has no singularities except perhaps at points ab.

This theorem gives an immediate result for a special class of Dirichlet series. For if we make the transformation $z = e^{-s}$ we see that the function defined by the series

$$\sum_{n=0}^{\infty} a_n b_n e^{-ns}$$

has singularities at most at the points $\alpha + \beta$, the points $\alpha = -\log a$ and $\beta = -\log b$ being the singularities of the functions defined by

$$\sum_{n=0}^{\infty} a_n e^{-ns}, \qquad \sum_{n=0}^{\infty} b_n e^{-ns}$$

respectively.

The following simple example § shows that this result does not hold when

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* Preliminary report presented to the American Mathematical Society, April 16, 1927.

† National Research Fellow in Mathematics.

‡ "Un théorème sur les séries de Dirichlet," *Comptes Rendus*, 1927, p. 1038.

§ J. Soula has called attention to this fact, making use of a slightly different example: *Journal de Mathématiques Pures et Appliquées*, Series 9, Vol. 4 (1925), p. 340.

these special Dirichlet series (power series) are replaced by general Dirichlet series of the type

$$(2) \qquad f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \qquad 0 \leqq \lambda_1 < \lambda_2 < \lambda_3 < \cdots, \quad \lim_{n=\infty} \lambda_n = \infty,$$

$$(3) \qquad \phi(s) = \sum_{n=1}^{\infty} b_n e^{-l_n s}, \qquad 0 \leqq l_1 < l_2 < l_3 < \cdots, \quad \lim_{n=\infty} l_n = \infty.$$

Take $a_n = b_n = 1$, $\lambda_n = l_n = \log n$. Then $f(s)$ and $\phi(s)$ are both equal to the function $\zeta(s)$ of Riemann,

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s.$$

This function is known to have no singularities in the finite plane except a pole at the point $s = 1$. But the function

$$\sum_{n=1}^{\infty} a_n b_n \, 1/n^s = \zeta(s)$$

has a singularity not at the point $\alpha + \beta = 2$.

We may, however, state Hadamard's theorem in a slightly different form, which indicates more clearly the nature of the situation for Dirichlet series. The function defined by

$$(1/1 - e^{-s}) \sum_{n=0}^{\infty} a_n e^{-ns} = \sum_{n=0}^{\infty} s_n e^{-ns}, \qquad s_n = \sum_{\nu=0}^{n} a_\nu$$

evidently has possible singularities at the points α and zero, so that the function defined by

$$\sum_{n=0}^{\infty} b_n s_n e^{-ns}$$

has possible singularities at the points $\alpha + \beta$ and β. It might then be supposed that the series

$$(4) \qquad \sum_{n=1}^{\infty} b_n s_n e^{-l_n s}$$

defines a function with possible singularities at the points $\alpha + \beta$ and β (α now being a singularity of $f(s)$ and β a singularity of $\phi(s)$). This result is clearly not contradicted by the example $f(s) = \phi(s) = \zeta(s)$ employed above. For, in this case the sum of the series (4) is $\zeta(s-1)$.

The series which we shall discuss

$$(5) \qquad \sum_{n=1}^{\infty} b_n e^{-l_n s} \sum_{\lambda_m < l_n} a_m (l_n - \lambda_m)^\rho$$

evidently reduces to (4) for $\rho = 0$. The method of proof employed demands, however, that ρ be always positive. But the position of the singularities of the function defined by (5) is as forecast by the above example. The discussion of the singularities of this function is made by use of the integral

$$(6) \qquad (1/2\pi i)\int_C f(s)\phi(z-s)\,(ds/s^{\rho+1})$$

suggested to the author by S. Mandelbrojt. It will be seen that this integral plays the rôle of the integral of Parseval in the proof of Hadamard's theorem.

2. *Proof of the Theorem.* We shall suppose that the series (2) converges for $\sigma > \sigma_1$, and that (3) converges absolutely for $\sigma > \sigma_2$ $(s = \sigma + it)$. Denote by α_j the singularities of $f(s)$ and by β_j those of $\phi(s)$, $(j = 1, 2, \cdots)$. We assume that the points α_j, β_j and $\gamma_j = \alpha_k + \beta_l$ are isolated. Furthermore, if

$$\alpha_j = \alpha_j' + i\alpha_j'',$$

we suppose that there exists a number r such that

$$(7) \qquad |\alpha_k' - \alpha_l'| > r \quad \text{when} \quad \alpha_k' \neq \alpha_l',$$

$$(8) \qquad |\alpha_k'' - \alpha_l''| > r \quad \text{when} \quad \alpha_k'' \neq \alpha_l'',$$

$$(9) \qquad |\beta_k - \beta_l| > r,$$

$$(10) \qquad |\gamma_k - \gamma_l| > r.$$

Under these conditions, the theorem which we wish to prove may be stated as follows.

If for an arbitrarily small number η, there is a $\mu > 0$ such that

$$f(s) = O(|t|^{\mu})^*$$

uniformly for $\sigma \geqq \pi_1$, $|s - \alpha_k| \geqq \eta$, and a $\nu > 0$ such that

$$\phi(s) = O(|t|^{\nu})$$

uniformly for $\sigma \geqq \pi_2$, $|s - \beta_k| \geqq \eta$, then the function defined† by the series

$$\sum_{n=1}^{\infty} b_n e^{-l_n s} \sum_{\lambda_m < l_n} a_m (l_n - \lambda_m)^{\rho}, \qquad \rho > \mu + \nu$$

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* For an explanation of this notation see G. H. Hardy and Marcel Riesz, *The General Theory of Dirichlet Series*, Cambridge (1915).

† If the function defined by (5) is not uniform the theorem applies to that branch of the function defined by the immediate prolongation of (5).

has no singularity, the real part of which is greater than the two numbers $\pi_1 + k_2$ *and* $\pi_2 + k_1$, *except perhaps the points* β_k *and* γ_k. *The quantities* k_1 *and* k_2 *are defined by the inequalities*

$$\alpha_k' \leqq k_1, \quad \beta_k' \leqq k_2, \qquad\qquad \beta_j = \beta_j' + i\beta_j''.$$

The demonstration of this theorem is based on a consideration of the integral (6) where the curve C is now to be described. Draw a line $\sigma = c_1$, where $c_1 = \pi_1$, if no point α nor the point zero lies on the line $\sigma = \pi_1$. Otherwise, choose c_1 so that no point α nor zero lies in the strip $\pi_1 < \sigma \leqq c_1$. This is possible by the condition (7). There is then a minimum distance δ from points α to the line $\sigma = c_1$. With each point α as center describe a small circle of radius r' as small as desired but less than δ and $r/2$. If the origin is not a point α, denote its distance to the nearest point α by δ_1. In this case make the radius r' of each of the above circles less than the least of the numbers δ, $r/2$, $\delta_1/_2$, and with the origin as center describe a circle of the same radius. The circles have been taken so small that no over-lapping will occur. Connect each of the circles lying to the right of $\sigma = c_1$ to this line by a regular curve in such a way that none of these lines cuts another nor any of the circles drawn. The curve C is to consist of the set of loops or "lacets" thus formed together with the line $\sigma = c_1$, and the sense of description is to be that of increasing t on the line and in the positive sense on the loops.

Now construct a curve C_β as follows. If s is a point on one of the loops described above, consider the locus of points

$$s + \beta_k \qquad\qquad (k = 1, 2, 3, \cdots).$$

The curve C_β is to consist of that part of this locus which lies to the right of the line

$$\sigma = c_1 + k_2 \qquad\qquad (\beta_k' \leqq k_2)$$

and of this line. Now construct a simply connected closed region D some points of which lie to the right of the lines $\sigma = \sigma_1 + \sigma_2$, $\sigma = \sigma_2$; all points of which lie to the right of the line $\sigma = k_1 + r' + \pi_2$, and no point of which is a point of C_β. If z lies in D we shall show that the improper integral

$$(11) \qquad F(z) = (1/2\pi i) \int_C f(s)\phi(z - s)\,(ds/s^{\rho+1}). \qquad\qquad \rho > \mu + \nu,$$

converges to a holomorphic function $F(z)$.

Denote the minimum distance from D to C_β by δ_2. Choose the number η

of the theorem less than δ_2 and less than r'. Then for s on C and z in D there exist numbers M and T independent of σ and z such that

$$(12) \qquad \begin{aligned} &|f(s)| < M|t|^{\mu}, && |t| \geqq T, \\ &|\phi(s)| < M|t|^{\nu}, && |t| \geqq T, \end{aligned}$$

for,

$$|s-\alpha_k| \geqq r' > \eta, \quad |z-s-\beta_k| \geqq \delta_2 > \eta \qquad \text{and}$$
$$\sigma \geqq \pi_1 + k_2, \quad x-\sigma \geqq \pi_2 + k_1, \qquad (z = x+iy).$$

Consequently the integral will converge uniformly in D if the integral

$$(13) \qquad \int_C \frac{|t|^{\mu+\nu}}{|s|^{\rho+1}} |ds|$$

converges. We consider first the integral over the line $\sigma = c_1$. That this converges may be easily seen by use of the equation

$$\int_{t_1}^{t_2} \frac{dt}{t^{\rho-\mu-\nu+1}} = \frac{1}{\rho-\mu-\nu}\left[\frac{1}{t_1^{\rho-\mu-\nu}} - \frac{1}{t_2^{\rho-\mu-\nu}}\right], \qquad \begin{aligned} &t_1 \neq 0, \quad t_2 \neq 0. \\ &\rho > \mu+\nu. \end{aligned}$$

On account of the condition (8) all the points α will surely lie on a denumerable set of lines $t = r_k$ where

$$r_{k+1} - r_k > r, \quad r_1 > 0, \quad r_{-1} < 0, \qquad k = 0, \pm 1, \pm 2, \cdots.$$

Hence for every point s on a loop about a point α which lies on $t = t_k$ we may suppose that $|s| > (|k|-1)r$, $|t| < |k| r$. If l denotes the maximum number of points α (for which $\alpha' > c_1$) on any line $t = r_k$ and L the maximum length of any loop, we see that the integral (13) extended over all the loops for which $|k| \geqq 2$ is less than

$$\frac{2lL}{r^{\rho-\mu-\nu+1}} \sum_{k=2}^{\infty} \frac{k^{\mu+\nu}}{(k-1)^{\rho+1}}$$

This series evidently converges if $\rho > \mu+\nu$. The uniform convergence of (11) is thus established.

We have only to apply a familiar theorem * concerning functions represented by definite integrals to recognize that $F(z)$ is analytic in D.

We shall now show that for x sufficiently large $F(z)$ may be represented by another integral (differing from (11) only in the path of integration) which can be expanded in a Dirichlet series. Let z be a point of D for which $x > \sigma_1 + \sigma_2$, $x > \sigma_2$ A number c_2 can then be chosen such that $c_2 > 0$,

* E. Goursat, *Cours d'analyse mathématique*, Paris (1911), Vol. 2, p. 268.

$c_2 > \sigma_1$ and $x - c_2 > \sigma_2$. For such points z it will now be possible to show that

$$F(z) = (1/2\pi i) \int_{c_2-i\infty}^{c_2+i\infty} f(s)\phi(z-s)\,(ds/s^{\rho+1}). \tag{14}$$

Here s and $z - s$ lie in the region of convergence of the series (2) and (3) respectively. Hence the proof of the convergence of (11) over $\sigma = c_1$ applies here. To prove the validity of (14) we make use of Cauchy's integral theorem.

Construct a rectangle whose vertices are the points

$$c_1 + ig_k, \quad c_2 + ig_k, \quad c_1 + ih_k, \quad c_2 + ih_k,$$
$$g_k = (r_k + r_{k+1})/2, \qquad h_k = (r_{-k} + r_{-k-1})/2.$$

Without loss of generality it may be assumed that the horizontal sides of the rectangle have no points in common with C. Denote by C_k the curve composed of the sides of this rectangle and those loops of C included in its interior. Then

$$\int_{C_k} f(s)\phi(z-s)\,(ds/s^{\rho+1}) = 0.$$

For, since $x - c_2 > \sigma_2$, and since all points α and the origin are exterior to the region bounded by C_k, the integrand is analytic there. If now we allow k to become infinite, the integrals

$$\int_{c_1+ig_k}^{c_2+ig_k} f(s)\phi(z-s)\,(ds/s^{\rho+1}), \qquad \int_{c_1+ih_k}^{c_2+ih_k} f(s)\phi(z-s)\,(ds/s^{\rho+1}),$$

approach zero by virtue of the conditions (12). The identity (14) then results immediately.

In order to expand the integral (14) we first note that the series

$$\sum_{n=1}^{\infty} [f(s)/s^{\rho+1}]\; b_n e^{-l_n(z-s)}$$

is uniformly convergent * on any finite line segment $\sigma = \sigma_2$, $-T \leqq t \leqq T$, (z being chosen as indicated above). It is consequently permissible to integrate term by term

$$(1/2\pi i) \int_{c_2-iT}^{c_2+iT} f(s)\phi(z-s)\,(ds/s^{\rho+1}) = \sum_{n=1}^{\infty} b_n e^{-l_n z}\; (1/2\pi i) \int_{c_2-iT}^{c_2+iT} [f(s)/s^{\rho+1}]\; e^{-l_n t}$$

* G. H. Hardy and M. Riesz, *loc. cit.*, p. 5.

If this series converges uniformly for $T \leqq T_0$, it will be permissible to allow T to become infinite term by term. This uniform convergence evidently results from the following inequalities:

$$\left| \int_{c_2-iT}^{c_2+iT} f(s) e^{ln s} (ds/s^{\rho+1}) \right| \leqq M e^{ln c_2} \int_{c_2-iT}^{c_2+iT} \frac{|t|^{\mu}}{|s|^{\rho+1}} |ds|$$

$$< M e^{ln c_2} \int_{c_2-i\infty}^{c_2+i\infty} \frac{|t|^{\mu}}{|s|^{\rho+1}} |ds| = K e^{ln c_2}$$

$$\sum_{n=1}^{\infty} b_n e^{-ln z} (1/2\pi i) \int_{c_1-iT}^{c_1+iT} [f(s)/s^{\rho+1}] e^{ln s} ds \ll (K/2\pi) \sum_{n=1}^{\infty} |b_n| e^{-ln(x-c_2)}.$$

The dominant series converges since (3) is assumed to converge absolutely for $\sigma > \sigma_2$.

If now we allow T to become infinite and make use of a familiar formula of J. Hadamard,* we see that

$$(15) \qquad F(z) = [1/\Gamma(\rho+1)] \sum_{n=1}^{\infty} b_n e^{-ln z} \sum_{\lambda_m < l_n} a_m (l_n - \lambda_m)^{\rho}.$$

The conditions of applicability of this formula are evidently satisfied since $c_2 > 0$, $c_2 > \sigma_1$.

This series expansion of $F(z)$ evidently applies to points z sufficiently far to the right in D. The integral expression (11) applies in the whole region and serves as an analytic prolongation of (15). On account of the arbitrary nature of the loops employed and by condition (10)† it follows that D may include any point of the plane not a point $\alpha + \beta$, β for which $x > \pi_1 + k_2$, $x > \pi_2 + k_1$. The theorem is thus completely established.

3. *Remarks.* If $\pi_1 = \pi_2 = -\infty$, the theorem evidently applies to the entire plane. That such a case may actually arise may be seen at once by use of power series (1) representing functions regular at infinity. An example of a function not defined by a power series and satisfying these conditions is the function

$$f(s) = \left(\sum_{n=1}^{\infty} 1/2^n n^s \right) \cdot \left(\sum_{n=1}^{\infty} e^{-ns} \right).$$

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* "Théorie générale des séries de Dirichlet," *Mémorial des sciences mathématiques*, Fasicule 17, p. 9.

† Without this condition the set γ might be dense even though (7), (8) and (9) be satisfied.

The first factor is a series absolutely convergent in all the plane, and the second is absolutely convergent for $\sigma > 0$. The product is consequently a Dirichlet series absolutely convergent for $\sigma > 0$. The function $f(s)$ surely has a singularity at $s = 0$, and we have $\mu = 0$, $\pi_1 = -\infty$.

As an example of the theorem take

$$f(s) = \zeta(s), \qquad \phi(s) = \sum_{n=1}^{\infty} e^{-ns}.$$

Here $\nu = 0$, $\pi_2 = -\infty$. The number π_1 may be taken negative and arbitrarily small. Then μ is any number greater than $\frac{1}{2} - \pi_1$.* It follows that the function defined by

$$\sum_{n=1}^{\infty} e^{-nz} \sum_{\log m < n} (n - \log m)^{\rho}$$

has no singularities to the right of the line $\sigma = \pi_1$ except perhaps the points $\beta_k = 2k\pi i$, $\alpha + \beta_k = 2k\pi i + 1$, $(k = 0, \pm 1, \pm 2, \cdots)$. In particular the series

$$\sum_{n=1}^{\infty} z^n \sum_{\log m < n} (n - \log m)$$

defines a function that has no singularities inside the circle $|z| \leqq 1$ except perhaps the points 1 and $1/e$. The theorem may thus be employed to obtain results about power series.

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* G. H. Hardy and M. Riesz, *loc. cit.*, p. 18.

On Entire Function Interpolation.

By I. M. Sheffer.

Introduction. To every entire function $f(z) = \sum_0^\infty c_n z^n$ we associate a function $P(q)$, $q > 0$, determined by the quantities $|c_0|, |c_1|, |c_2|, \cdots, |c_n|, \cdots$. The definition of $P(q)$ is given in § 1. In this section some simple properties of $P(q)$ are proved, and application is made to prove that *if* a_n $(n = 0, 1, \cdots)$ is a given set of numbers, there exist infinitely many entire functions $E(z)$ such that $E(n) = a_n$. This theorem is usually proved by means of a theorem due to Mittag-Leffler.* The proof given here, however, follows from the properties of $P(q)$ in a very elementary manner.

In § 2 it is shown that the first k derivatives (k arbitrary) of $E(z)$ may also be assigned. The proof is by induction. In § 3 application of the final result of § 1 is made to give a new proof of a theorem due to Borel.† We are led to an interesting set of polynomials of which we state some important properties.

1. *The Function $P(q)$, and an Application.* Let $f(z) = \sum_0^\infty c_n z^n$ be entire. To each $q > 0$ corresponds a positive number $P(q)$ such that

$$|c_n| \leqq P(q)q^n, \qquad (n = 0, 1, \cdots) \tag{1}$$

with the equality sign holding for at least one value of n. It is clear that the equality sign can hold only for a finite number of values of n.

Definition. The values of n for which $|c_n| = P(q)q^n$ shall be termed *maximum indices* corresponding to q.

We now state some simple properties of maximum indices:

Lemma 1.‡ 1) If $n = n_1$, n_2 are the smallest maximum indices for $q = q_1$, q_2 respectively, where $q_2 < q_1$, then $n_2 \geqq n_1$.

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* Osgood, *Funktionentheorie*, Vol. 1, 4th edition, Leipzig (1923), pp. 539-551.

† Borel, *Leçons sur les Séries Divergentes*, Paris (1901), pp. 152-153.

‡ The proofs of 1) to 8) are of the same nature, so it suffices to prove 1): Consider

$$|c_0|/{q_1}^0, \ldots, |c_{n_1}|/{q_1}^{n_1} \text{ and } |c_0|/{q_2}^0, \ldots, |c_{n_1}|/{q_2}^{n_1}.$$

By hypothesis $|c_{n_1}|/{q_1}^{n_1} > |c_i|/{q_1}^i$, $(i = 0, 1, \ldots, n_1 - 1)$. Let $q_2 = q_1 a$. Then $a < 1$, and

$$|c_{n_1}|/{q_2}^{n_1} = |c_{n_1}|/(aq_1)^{n_1} > (|c_i|/{q_1}^i)(1/a^{n_1}) > |c_i|/(aq_1)^i = |c_i|/{q_2}^i, \quad (i = 0, 1, \ldots, n_1 - 1).$$

Hence the first maximum index for $q = q_2$ is $\geqq n_1$.

2) If c_s is the first non-zero Taylor coefficient of $f(z)$, then for all q sufficiently large s is the only maximum index. (If $c_0 \neq 0$, q' exists such that $P(q) = \text{constant}$, $q > q'$.)

3) If to $q = q_0$ there correspond the maximum indices $n_0, n_1, \cdots, n_k$, (in order of increasing magnitude), then:

i) for $q < q_0$ no one of the indices $n_0, n_1, \cdots, n_{k-1}$ is a maximum index; and

ii) for $q > q_0$ no one of the indices $n_1, \cdots, n_k$ is a maximum index.

4) If n_0 is a maximum index for q_0, there exists a new $q = q'$, $\leqq q_0$ such that for $q < q'$, n_0 is no longer a maximum index.

5) If n_0 is the only maximum index for q', then q_0, q_1 exist, $q_0 < q' < q_1$, such that n_0 is the only maximum index for each q in $q_0 < q < q_1$.

6) If n_0, n_k are respectively the first and last maximum indices for q_0, then q_1, q_2 exist, $q_2 < q_0 < q_1$, such that:

i) n_0 is the only maximum index, $q_0 < q < q_1$;

ii) n_k is the only maximum index, $q_2 < q < q_0$.

7) If n_0 is the only maximum index for $q_0 < q < q_1$, then n_0 is a maximum index for $q = q_0, q_1$.

8) $P(q_1) \geqq P(q_2)$, $0 < q_1 < q_2$.

LEMMA 2. The function $P(q)$ is continuous, $0 < q$.

Proof: Case i) n_0 is the only maximum index for $q = q_0$. The lemma then follows from 5), since $P(q) = |c_{n_0}|/q^{n_0}$ in the neighborhood of q_0. Case ii): There is more than one maximum index for q_0. Let n_0 be the first and n_k the last. By 6), we can find $q_1 > q_0$, $q_2 < q_0$ such that n_0 is the only maximum index, $q_0 < q < q_1$, and n_k is the only maximum index, $q_2 < q < q_0$. Therefore $P(q)$ is continuous in $q_0 < q < q_1$, and in $q_2 < q < q_0$. Also, $P(q_0^+) = P(q_0^-) = P(q_0)$. Therefore $P(q)$ is continuous at q_0.

LEMMA 3. $P'(q)$ is continuous at each $q > 0$ for which there is a single maximum index, and $P'(q^+)$, $P'(q^-)$ exist (but are unequal) at each $q > 0$ for which there are more maximum indices than one.

Proof: i) If for q_0 there is a single maximum index, n_0, then in the neighborhood of q_0, $P(q) = |c_{n_0}|/q^{n_0}$; whence $P'(q)$ is continuous at q_0.

ii) Let there be more than one maximum index corresponding to q_0. By 6), $P'(q)$ is continuous in the deleted * neighborhood of q_0, and †

$$\lim_{q=q_0^+} P'(q) = P'(q_0^+) = -P(q_0)n_0/q_0;$$

$$\lim_{q=q_0^-} P'(q) = P'(q_0^-) = -P(q_0)n_k/q_0.$$

Since the first maximum index never decreases as $q > 0$ *decreases,* we see that $P(q)$ represents a continuous curve having a derivative at every point with the exception of a denumerable infinity of angular points.‡ An angular point occurs when and only when there corresponds to q more than one maximum index.

Denote the angular points by $q = \xi_0, \xi_1, \cdots, \xi_n, \cdots$, in order of *decreasing* magnitude. Clearly the only cluster point is $q = 0$. Since the definition of $P(q)$ involves only the absolute values of the c_n's, it follows that there are infinitely many entire functions having the same P-function.

Given an arbitrary set of constants ξ_n satisfying the single condition $\xi_0 > \xi_1 > \cdots > \xi_n > \cdots$, $\lim \xi_n = 0$; and an arbitrary set of integers k_n satisfying $0 \leqq k_0 < k_1 < \cdots < k_n < \cdots$. Define a function $Q(q)$, $0 < q$ as follows:

$$(2) \qquad \begin{aligned} &Q(q) = \alpha/q^{k_0}, \quad \alpha > 0, \quad \xi_0 \leqq q; \\ &Q(\xi_n) = (\xi_{n-1}/\xi_n)^{k_n} Q(\xi_{n-1}); \\ &Q(q) = (\xi_n/q)^{k_n} Q(\xi_n), \quad \xi_n < q < \xi_{n-1}. \end{aligned}$$

We see that $Q(q)$ is positive and continuous, and that $Q'(q)$ is continuous at every $q \neq \xi_0, \xi_1, \cdots$, while $Q'(q^+)$, $Q'(q^-)$ exist (and are unequal) at every point $q = \xi_0, \xi_1, \cdots$. Furthermore, $Q(q_1) > Q(q_2)$, $0 < q_1 < q_2 \leqq \xi_0$.

LEMMA 4. There exists an entire function (and consequently an infinite number of them) whose P-function has the points $q = \xi_0, \xi_1, \cdots$ (and only those) as angular points, and such that in the interval $\xi_n < q < \xi_{n-1}$, k_n is the only maximum index.§ Furthermore, the P-function for this entire function is precisely $Q(q)$.

Proof: $Q(q)q^{k_n} = \xi^{k_n}_{n-1} Q(\xi_{n-1}) =$ a positive constant, $\xi_n \leqq q \leqq \xi_{n-1}$;

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* That is, $q = q_0$ itself is excepted.

† n_0, n_k are respectively the first and last maximum indices for $q = q_0$.

‡ If $f(z)$ is a polynomial there are only a finite number of angular points.

§ For $q > \xi_0$, k_0 is the only maximum index.

denote this constant by * A_{k_n}. Consider the function $f(z) = \sum_{n=0}^{\infty} A_{k_n} z^{k_n}$. We have

$$\frac{Ak_{n+1}}{Ak_n} = \xi_n^{k_{n+1}-k_n}(\xi_n/\xi_{n-1})^{k_n}\,\frac{Q(\xi_n)}{Q(\xi_{n-1})} = \xi_n^{k_{n+1}-k_n}(\xi_n/\xi_{n-1})^{k_n}\quad(\xi_{n-1}/\xi_n)^{k_n} = \xi_n^{k_{n+1}-k_n}.$$

Therefore $\lim_{n=\infty} \frac{Ak_{n+1}z^{k_{n+1}}}{Ak_n\, z^{k_n}} = \lim_{n=\infty} (z\,\xi_n)^{k_{n+1}-k_n} = 0$, since $k_{n+1} - k_n \geqq 1$ and $\xi_n \to 0$. Hence $f(z)$ is an entire function.† We wish to show that its P-function is $Q(q)$.

Let q be any point of $\xi_n < q < \xi_{n-1}$, and consider

$$\{\quad\} = \{(Ak_p/q^{k_p})/(Ak_n/q^{k_n})\} = q^{k_n-k_p}\,\frac{Ak_p}{Ak_n}, \qquad p \neq n.$$

Now

$$\frac{Ak_{n+1}}{Ak_n} = \xi_n^{k_{n+1}-k_n}, \qquad \frac{Ak_{n+2}}{Ak_n} = \xi_{n+1}^{k_{n+2}-k_{n+1}}\cdot\,\xi_n^{k_{n+1}-k_n},$$

and in general,

$$\frac{Ak_{n+s}}{Ak_n} = \xi_{n+s-1}^{k_{n+s}-k_{n+s-1}}\cdot\,\xi_{n+s-2}^{k_{n+s-1}-k_{n+s-2}}\cdot\cdot\cdot\,\xi_n^{k_{n+1}-k_n}.$$

Similarly,

$$\frac{Ak_{n-s}}{Ak_n} = \left[\,\xi_{n-1}^{k_n-k_{n-1}}\cdot\,\xi_{n-2}^{k_{n-1}-k_{n-2}}\cdot\cdot\cdot\,\xi_{n-s}^{k_{n-s+1}-k_{n-s}}\,\right]^{-1}.$$

Hence if $p > n$: $p = n + s$, then

$$\{\quad\} = q^{k_n-k_p}\,\xi_{n+s-1}^{k_{n+s}-k_{n+s-1}}\cdot\cdot\cdot\,\xi_n^{k_{n+1}-k_n} \leqq q^{k_n-k_p}\,\xi_n^{[(k_{n+s}-k_{n+s-1})+\,\cdots\,+(k_{n+1}-k_n)]}$$
$$= q^{k_n-k_p}\,\xi_n^{k_p-k_n} = (\xi_n/q)^{k_p-k_n} < 1$$

since $q > \xi_n$ and $k_p > k_n$.

And if $p < n$: $p = n - s$, then

$$\{\quad\} = q^{k_n-k_p}\left[\xi_{n-1}^{k_n-k_{n-1}}\cdot\cdot\cdot\,\xi_{n-s}^{k_{n-s+1}-k_{n-s}}\right]^{-1} \leqq q^{k_n-k_p}\,\xi_{n-1}^{k_n-k_{n-s}} = (q/\xi_{n-1})^{k_n-k_p} < 1.$$

Consequently k_n is the only maximum index for $\xi_n < q < \xi_{n-1}$. From 7), then, k_n is a maximum index for $q = \xi_{n-1}$, ξ_n. Therefore k_{n+1}, k_n are maximum indices for $q = \xi_n$, and k_n, k_{n-1} for $q = \xi_{n-1}$. This shows that the angular points of $P(q)$ are precisely $q = \xi_0, \xi_1, \cdots$. There remains to identify P with Q.

By definition of a maximum index we have $A_{k_n} = P(q)q^{k_n}$, and this holds for $\xi_n \leqq q \leqq \xi_{n-1}$, $n \geqq 1$. Comparing this with the value of $Q(q)$ we have

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* $A_{k_0} = Q(\xi_0)\xi_0^{k_0} = a$.

† $f(z)$ is not a polynomial.

$P(q) = Q(q)$, $\xi_n \leqq q \leqq \xi_{n-1}$, $n \geqq 1$. That is, $P(q) = Q(q)$, $q \leqq \xi_0$. From the property of the P-function, k_0 (A_{k_0} being the first non-zero coefficient of $f(z)$) is the only maximum index for $q > \xi_0$. Therefore $P(q) = A_{k_0}/q^{k_0}$, $q \geqq \xi_0$. That is, $P(q) = [\xi_0^{k_0} Q(\xi_0)]/q^{k_0} = \alpha/q^{k_0} = Q(q)$, $q \geqq \xi_0$. Then, $P(q) = Q(q)$, $0 < q$. The proof is now complete.

Define $F(r)$ by

$$F(r) = \sum_0^\infty | c_n | r^n. \tag{3}$$

LEMMA 5. We have

$$F(r) < P(q) \sum_0^\infty (rq)^n = P(q)/(1-rq), \quad rq < 1. \tag{4}$$

The proof is immediate.

LEMMA 6. We have

$$F(r) > P(1/r), \quad r > 0. \tag{5}$$

Proof: From (3), $F(r) > | c_n | r^n$, $n = 0, 1, \cdots$. Let n_0 be a maximum index for any value q: $| c_{n_0} | = P(q) q^{n_0}$. Then $F(r) > P(q)(rq)^{n_0}$. (5) now follows on setting $q = 1/r$.

If in (4) we set $q = 1/(r+1)$ we obtain the following corollary:

Corollary. The Function P satisfies the inequality

$$P(1/r) < F(r) < (r+1)P(1/r+1), \quad r > 0. \tag{6}$$

LEMMA 7. Let α_n be an arbitrary set of numbers, $n = 0, 1, \cdots$. There exists an entire function $K(z)$ (and consequently an infinite number) such that $K(n) > | \alpha_n |$.

Proof: Set * $A_n = 1 + | \alpha_{n+1} |$, and choose $\xi_{n-1} = 1/n$, $n = 1, 2, \cdots$. Finally, choose the set of integers k_n: $0 \leqq k_0 < k_1 < \cdots$ such that †

$$\begin{aligned} B_1 &= (\xi_0/\xi_1)^{k_1} \geqq A_1 \\ B_2 &= (\xi_1/\xi_2)^{k_2} A_1 \geqq A_2 \\ &\cdots\cdots\cdots\cdots \\ B_n &= (\xi_{n-1}/\xi_n)^{k_n} A_{n-1} \geqq A_n \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{7}$$

By Lemma 4, there exists an entire function $f(z) = \sum_0^\infty c_n z^n$, $c_n \geqq 0$, cor-

* We deal with A_n rather than with α_n because some of the α_n's may be zero.

† This choice can be made, since $\xi_n < \xi_{n-1}$.

responding to the ξ_n's and k_n's. Consider the P-function corresponding to $f(z)$:

$$P(q) = \xi^{k_n}{}_{n-1}\, P(\xi_{n-1}) q^{-k_n}, \quad \xi_n \leqq q \leqq \xi_{n-1}.$$

Setting $q = \xi_n$: $P(\xi_n) = (\xi_{n-1}/\xi_n)^{k_n} P(\xi_{n-1})$, $n \geqq 1$, and $P(\xi_0) = \alpha/\xi_0^{k_0}$, $= 1$ by proper choice of α. Then

$$\begin{aligned} P(\xi_1) &= (\xi_0/\xi_1)^{k_1} = B_1 \geqq A_1 \\ P(\xi_2) &= (\xi_1/\xi_2)^{k_2} P(\xi_1) \geqq (\xi_1/\xi_2)^{k_2} A_1 = B_2 \geqq A_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P(\xi_n) &= (\xi_{n-1}/\xi_n)^{k_n} P(\xi_{n-1}) \geqq (\xi_{n-1}/\xi_n)^{k_n} A_{n-1} = B_n \geqq A_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{8}$$

Now $\xi_n = 1/n + 1$. Consequently

$$P(1/n+1) \geqq A_n > |\, \alpha_{n+1}\,|, \quad n = 1, 2, \cdots.$$

Using Lemma 6, $f(n+1) > P(1/n+1) > |\, \alpha_{n+1}\,|$, $n = 1, 2, \cdots$. Then $K(z) = |\, \alpha_0\,| + |\, \alpha_1\,| + f(z)$ is entire, and $K(n) > |\, \alpha_n\,|$, $n = 0, 1, \cdots$.

LEMMA 8. If $|\, c_n\,| \leqq 1/(n+1)^2$, $n = 0, 1, \cdots$, then

$$\omega(z) = \sum_0^\infty c_n \frac{\sin \pi(z-n)}{\pi(z-n)} \tag{9}$$

is an entire function, and $\omega(n) = c_n$.

Proof: Consider the region $|\, z\,| \leqq R$, R arbitrary but fixed. Neglect those terms (say $n = 0, 1, \cdots, n_0 - 1$) for which $n - R < 1$. Then, writing $z = x + iy$,

$$\sum_{n=n_0}^\infty \left| c_n \frac{\sin \pi(z-n)}{\pi(z-n)} \right| \leqq \sum_{n_0}^\infty \frac{1}{\pi(n+1)^2} \left| \frac{e^{i\pi(z-n)} - e^{-i\pi(z-n)}}{2i} \right|$$

$$\leqq \sum_{n_0}^\infty \frac{1}{2\pi(n+1)^2} (e^{\pi y} + e^{-\pi y}) \leqq \sum_{n_0}^\infty \frac{e^{\pi R}}{\pi(n+1)^2}.$$

That is, $\omega(z)$ is analytic in $|\, z\,| \leqq R$, R arbitrary. Hence $\omega(z)$ is entire. Setting $z = n$: $\omega(n) = c_n$.

Corollary. The function

$$V(z) = (z+1)^2 \sum_0^\infty \frac{1}{(n+1)^2} \frac{\sin \pi(z-n)}{\pi(z-n)} \tag{10}$$

is entire, and * $V(n) = 1$, $n = 0, 1, \cdots$.

* But $V(z) \not\equiv 1$.

We now arrive at the desired theorem:

THEOREM 1. *Given an arbitrary set of numbers* a_n, $n = 0, 1, \cdots$. *There exist infinitely many entire functions* $E(z)$ *for which* $E(n) = a_n$.

Proof: It suffices to find one such function, since we can then multiply it by any positive power of $V(z)$ of equation (10) to get further such functions. By Lemma 7, $K(z)$, entire, exists such that $b_n = K(n) > (n+1)^2 |\, a_n \,|$. Set $c_n = a_n/b_n$. Then $|\, c_n \,| \leqq 1/(n+1)^2$, whence from Lemma 8, an entire function $\omega(z)$ exists such that $\omega(n) = c_n$. Therefore $E(z) = K(z)\omega(z)$ is entire, and $E(n) = a_n$.

Before taking up the work of § 2, we remark that the problem just solved is completely equivalent * to the problem of finding a solution of the system of linear equations

$$\sum_{j=0}^{\infty} i^j x_j = a_i, \quad i = 0, 1, \cdots. \tag{11}$$

2. $E(n)$, $E'(n)$, $\cdots$, $E^{(k)}(n)$ *Assigned,* $n = 0, 1, \cdots$. In this section we shall be concerned with the proof of

THEOREM 2. *Given* $(k+1)$ sets of numbers ${}_0c_n, {}_1c_n, \cdots, {}_kc_n$, $n = 0, 1, \cdots$. *Infinitely many entire functions* $E(z)$ *exist for which* $E^{(r)}(n) = {}_rc_n$, $r = 0, 1, \cdots, k$.

The proof will be carried through by induction.

LEMMA 9. Let c_n be any set of numbers. Infinitely many entire functions $G(z)$ exist such that $G(n) = c_n$, $G'(n) = 0$, $n = 0, 1, \cdots$.

Proof: By Theorem 1 there exists $F(z)$, entire, such that $F(n) = c_n$. Consider the entire function $\zeta(z) = z - 1 + \cos 2\pi z - (1/2\pi) \sin 2\pi z$. We have $\zeta(n) = n$, $\zeta'(n) = 0$, $n = 0, 1, \cdots$. Set † $G(z) = F(\zeta(z))$. $G(z)$ is entire, and $G'(z) = \dfrac{dF(\zeta)}{d\zeta} \dfrac{d\zeta}{dz}$. Therefore $G(n) = c_n$, $G'(n) = 0$. $G(z)$ is then a solution.

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* That is, every solution $E(z) = \sum_{i=0}^{\infty} x_i z^i$ of § 1 gives a solution (x_i) of (11), and conversely.

† There are clearly an infinity of such functions $G(z)$, since this is true of $F(z)$. This remark applies to the later lemmas also.

LEMMA 10. Let c_n, e_n be any two sets of numbers. There exist infinitely many entire functions $G(z)$ such that $G(n) = c_n$, $G'(n) = e_n$, $n = 0, 1, \cdots$.

Proof: By Theorem 1 and Lemma 9, entire functions $A(z)$, $B(z)$, $C(z)$ exist * such that

$$A(n) = c_n, \quad A'(n) = 0; \quad B'(n) = e_n; \quad C(n) = B(n), \quad C'(n) = 0.$$

Then $G(z) = A(z) + B(z) - C(z)$ satisfies the conditions of the lemma.

This proves Theorem 2 for the case $k = 1$. We now turn to the general case. *Assume* the theorem to hold for $k - 1$. We shall then prove it for k.

LEMMA 11. Given c_n; infinitely many entire functions $G(z)$ exist such that $G(n) = c_n$, $G^{(r)}(n) = 0$, $r = 1, 2, \cdots, k$.

Proof: By hypothesis, $F(z)$ exists such that $F(n) = c_n$, $F^{(r)}(n) = 0$, $r = 1, \cdots, k-1$. Let $\theta(z)$ be an entire function, which we shall presently determine, and consider the function

$$G(z) = F(\theta(z)). \tag{12}$$

We have

$$\begin{aligned} G(z) &= F(\theta) \\ G'(z) &= \frac{dF(\theta)}{d\theta}\ \frac{d\theta}{dz} \\ G''(z) &= \frac{dF(\theta)}{d\theta}\ \frac{d^2\theta}{dz^2} + \frac{d^2F(\theta)}{d\theta^2}\left(\frac{d\theta}{dz}\right)^2 \qquad (13) \\ G'''(z) &= \frac{dF(\theta)}{d\theta}\ \frac{d^3\theta}{dz^3} + \frac{d^3F(\theta)}{d\theta^3}\left(\frac{d\theta}{dz}\right)^3 + 3\,\frac{d^2F(\theta)}{d\theta^2}\ \frac{d\theta}{dz}\ \frac{d^2\theta}{dz^2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

In general, $G^{(r)}(z)$ contains a term $\frac{dF(\theta)}{d\theta}\ \frac{d^r\theta}{dz^r}$, and a term $\frac{d^rF(\theta)}{d\theta^r}\left(\frac{d\theta}{dz}\right)^r$; all the other terms † contain some derivative of $F(\theta)$ with respect to θ, of order s, $2 \leqq s \leqq r-1$. Choose

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* If $H(z)$, entire, is such that $H(n) = e_n$, then $B(z) = \int_0^z H(z)\,dz$ is entire and its derivative takes on the values e_n.

† This is true for $r = 1, 2, 3$. Assume it true up to $r-1$:

$G^{(r-1)}(z) = \frac{dF(\theta)}{d\theta}\ \frac{d^{(r-1)}\theta}{dz^{r-1}} + \frac{d^{r-1}F(\theta)}{d\theta^{r-1}}\left(\frac{d\theta}{dz}\right)^{r-1}$ + terms involving derivatives of $F(\theta)$ with respect to θ of degree $\leqq r-2$. On differentiating with respect to z, we get the result stated.

$$\theta(z) = \zeta(z) + \omega_1(z) + \cdots + \omega_{k-2}(z), \tag{14}$$

where $\zeta(z) = z - 1 + \cos 2\pi z - (1/2\pi)\sin 2\pi z$, and $\omega_i(z)$, entire, is defined as follows: *

$\omega_1(z) = A_1 (\sin 2\pi z)^3$, choosing A_1 so that
$\omega_1(n) = \omega_1'(n) = \omega_1''(n) = 0, \quad \omega_1^{(3)}(n) = -\zeta^{(3)}(n).$
$\omega_2(z) = A_2 (\sin 2\pi z)^4$, choosing A_2 so that
$\omega_2(n) = \omega_2'(n) = \omega_2''(n) = \omega_2^{(3)}(n) = 0,$
$\omega_2^{(4)}(n) = -[\zeta^{(4)}(n) + \omega_1^{(4)}(n)].$

. . . . . .

$\omega_{k-2}(z) = A_{k-2} (\sin 2\pi z)^k$, choosing A_{k-2} so that
$\omega_{k-2}(n) = \omega'_{k-2}(n) = \cdots = \omega_{k-2}^{(k-1)}(n) = 0,$
$\omega_{k-2}^{(k)}(n) = -[\zeta^{(k)}(n) + \omega_1^{(k)}(n) + \cdots \omega_{k-3}^{(k)}(n)].$

Such a choice of $A_1, \cdots, A_{k-2}$ can be made since $\zeta^{(r)}(n)$ and $\omega_i^{(r)}(n)$ $(i = 1, \cdots, r-3)$ are independent of n. Then $\theta(z)$ is entire and $\theta(n) = \zeta(n) = n$, $\theta'(n) = 0$, $\theta''(n) = -4\pi^2$, $\theta^{(r)}(n) = 0$, $r = 3, 4, \cdots, k$.

Now $\left.\dfrac{d^rF(z)}{dz^r}\right|_{z=n} = 0$, $r = 1, \cdots, k-1$, and since $\theta(n) = n$, it follows that $\left.\dfrac{d^rF(\theta)}{d\theta^r}\right|_{z=n} = 0$, $r = 1, \cdots, k-1$. Hence, returning to (13):

$$G(n) = F(\theta(n)) = c_n.$$

$$G'(n) = \left.\frac{dF(\theta)}{d\theta}\right|_{z=n} \cdot \theta'(n) = 0$$

. . . . . .

$$G^{(r)}(n) = \left.\frac{dF(\theta)}{d\theta}\right|_{z=n} \cdot \theta^{(r)}(n) + \left.\frac{d^rF(\theta)}{d\theta^r}\right|_{z=n} \cdot [\theta'(n)]^r +$$

[terms involving derivatives of $F(\theta)$ with respect to θ of order $\leqq r-1$]

$$= 0, \quad r = 2, 3, \cdots, k.$$

The lemma is now proved.

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* In this definition we make use of the fact that at the points $z = n$, $(\sin 2\pi z)^r$ and its first $r-1$ derivatives are zero, the rth derivative is not zero, and all derivatives after the $(r-1)$st are independent of n.

Now let ${}_0c_n, \cdots, {}_kc_n$ be assigned. By Lemma 11 and our assumption for $k-1$, we can find $A(z)$, $B(z)$, $C(z)$ such that

$$A(n) = {}_0c_n,\ A'(n) = {}_1c_n,\ \cdots,\ A^{(k-1)}(n) = {}_{k-1}c_n;$$
$$^*B'(n) = 0, \cdots, B^{(k-1)}(n) = 0,\ B^{(k)}(n) = {}_kc_n - A^{(k)}(n);$$
$$C(n) = B(n),\ C'(n) = 0, \cdots, C^{(k-1)}(n) = 0,\ C^{(k)}(n) = 0.$$

Then $E(z) = A(z) + B(z) - C(z)$ is entire and $E^{(r)}(n) = {}_rc_n$, $r = 0, 1, \cdots, k$. There are infinitely many such functions $E(z)$.† The proof of Theorem 2 is now complete.

Remarks. 1) Theorem 2 is equivalent to the theorem:

THEOREM. *There are infinitely many solutions* (x_i) *of the* $(k+1)$ *simultaneous systems*

$$\sum_{j=0}^{\infty} i^j x_j = {}_0c_i,\quad \sum_{j=0}^{\infty} j i^{j-1} x_j = {}_1c_i,\ \cdots,\ \sum_{j=0}^{\infty} j(j-1)\cdots(j-k+1) i^{j-k} x_j = {}_kc_i,$$
$$i = 0, 1, \cdots.$$

2) If ${}_0c_n, \cdots, {}_kc_n$ are real, then infinitely many of the solutions $E(z)$ are real; i. e., if $E(z) = \sum_0^\infty g_n z^n$, then g_n is real.

3) There are infinitely many solutions $E(z) = \sum_0^\infty g_n z^n$ for which $g_0, g_1, \cdots, g_{r-1}$ (r arbitrary) can be preassigned, provided we do not give ${}_0c_n, {}_1c_n, \cdots, {}_kc_n$ for $n = 0$.

Proof: 1) is immediate; 2) follows from consideration of the proof of Theorems 1 and 2; 3) follows from Theorem 2 by choosing‡ ${}_ic_0 = i!\, g_i$, $i = 0, 1, \cdots, r-1$.

3. *Proof of a Theorem of Borel.* Consider the function

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots, \tag{1}$$

analytic about the origin. An entire function

$$\zeta(z) = b_0 + b_1 z + \cdots + b_n z^n + \cdots \tag{2}$$

exists such that $\zeta(n) = a_n$, $n = 0, 1, \cdots$. That is,

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* See footnote (*), page 336.

† See footnote (†), page 335.

‡ If $r-1 > k$, we can assign ${}_ic_n$ at pleasure $(n \geqq 1)$, $(i = k+1, \ldots, r-1)$, and get a solution.

$$a_n = b_0 + nb_1 + n^2 b_2 + \cdots + n^k b_k + \cdots. \tag{3}$$

Therefore

$$f(z) = b_0 + z\left(\sum_0^\infty b_k 1^k\right) + z^2\left(\sum_0^\infty b_k 2^k\right) + \cdots. \tag{4}$$

If we combine *formally* the coefficients of b_k and sum from $n = 0$ to ∞ we obtain

$$f(z) \sim b_0 u_0(z) + b_1 u_1(z) + \cdots + b_n u_n(z) + \cdots, \tag{5}$$

where

$$\begin{aligned} u_0(z) &= 1 + z + z^2 + \cdots + z^k + \cdots; \\ u_n(z) &= z + 2^n z^2 + \cdots + k^n z^k + \cdots, \quad n > 0. \end{aligned} \tag{6}$$

We proceed to discuss the functions $u_n(z)$. Clearly

$$\begin{aligned} u_0(z) &= 1/(1-z), \\ u_n(z) &= z u'_{n-1}(z). \end{aligned} \tag{7}$$

The first few $u_n(z)$'s are

$$u_0(z) = 1/(1-z), \quad u_1(z) = z/(1-z)^2, \quad u_2(z) = z(1+z)/(1-z)^3.$$

This suggests that we can write $u_n(z)$ in the form

$$u_n(z) = P_n(z)/(1-z)^{n+1}, \tag{8}$$

where $P_n(z)$ is a polynomial of degree precisely n. To prove this we *define* $P_n(z)$ by (8) and substitute into (7). On simplifying we see that

$$\begin{aligned} P_0(z) &= 1, \\ P_n(z) &= z(1-z)P'_{n-1}(z) + nzP_{n-1}(z). \end{aligned} \tag{9}$$

From (9) we obtain (by induction) the following properties of $P_n(z)$:

1) $P_n(z)$ is a real polynomial of degree n.
2) $P_n(0) = 0$, $n > 0$; $P_n(1) = n!$
3) The coefficients of $P_n(z)$ are positive * integers.
4) $P_n(x) > 0$, $x > 0$; $|P_n(z)| \leqq n!$, $|z| \leqq 1$.

Further properties are:

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* If we set $P_n(z) = a_{n,n}z^n + \ldots + a_{n,0}$ and substitute into (9), the relations defining $a_{n,i}$ in terms of the $a_{n-1,k}$'s are linear with *positive* coefficients. Of course $P_n(z)$ has no constant term, $n > 0$.

5) $P_n(x_0) > x_0^n n!$, $0 < x_0 < 1$.

Proof: $P_n(x_0) > 0$, $P_{n-1}(x_0) > 0$, $P'_{n-1}(x_0) > 0$,

$$x_0(1-x_0)P'_{n-1}(x_0) > 0.$$

Therefore

$$P_n(x_0) > nx_0P_{n-1}(x_0) > nx_0$$
$$[(n-1)x_0P_{n-2}(x_0)] > \cdots > n(n-1)\cdots 2x_0^{n-1}P_1(x_0) = n!x_0^n.$$

6) $P_n(x_0) < x_0^n n!$, $x_0 > 1$.

The proof is like that of 5).

Corollary. We have $|P_n(z)| \leqq n!|z|^n$, $|z| > 1$.

LEMMA 12. If $\sum_0^\infty |c_n u_n(x_0)|$ converges, where x_0 is real and satisfies $(1/2) \leqq x_0 < 1$, then M exists such that $|c_n| \leqq M/n!$ for all n.

Proof: We have $u_n(x_0) = P_n(x_0)/(1-x_0)^{n+1} > 0$, and $P_n(x_0) > n!x_0^n$. Therefore

$$\sum_0^\infty |c_n| u_n(x_0) \gg [1/(1-x_0)] \sum_0^\infty r^n n!,$$

where $r = x_0/(1-x_0) \geqq 1$. By hypothesis the series on the right must converge, and from this fact the lemma easily follows.

Corollary. Under the same hypothesis, $|c_n| \leqq (M/n!)[(1-x_0)/x_0]^n$.

LEMMA 13. If $|c_n| \leqq (M/n!)$, then $f(z) = \sum_0^\infty c_n u_n(z)$ converges absolutely for every (finite) z in the region * T: $R(z) < 1/2$, $|1-z| > 1$; and uniformly in every finite (closed) region in T. Consequently $f(z)$ is analytic in T.

Proof: Let z be any point in T.

i) $|z| \geqq 1$. Then $|P_n(z)| \leqq n!|z|^n$, from which follows that $\sum_0^\infty |c_n u_n(z)| \leqq (M/|1-z|) \sum_0^\infty |z/(1-z)|^n$, which converges since $|z/(1-z)| < 1$ in T.

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* $R(z)$ = real part of z.

ii) $|z| \leqq 1$. Then $|P_n(z)| \leqq n!$, and $\sum_0^\infty |c_n u_n(z)| \leqq M/|1-z| \sum_0^\infty 1/|1-z|^n$, which converges.

This proves absolute convergence. Now let S be any (finite) closed region in T. Divide S into two closed parts S_1, S_2 (if necessary) by the circle $|z| = 1$. Uniform convergence in S_1 and in S_2 follow at once, and therefore in S.

Let us now return to the function $f(z)$.

THEOREM 3. *If $f(z) = \sum a_n z^n$ is analytic about the origin, and if an entire function $\zeta(z) = \sum_0^\infty b_n z^n$ exists with $|b_n| \leqq M/n!$, such that $\zeta(n) = a_n$, then $f(z)$ is analytic at every (finite) point in T.*

Proof: It is easily * seen that equation (5) of § 3 is valid in the neighborhood of the origin. Now apply Lemma 13.

Corollary 1. If $\zeta(z) = \sum_0^\infty b_n z^n$ is entire with $|b_n| \leqq M/n!$, then $F(z) = \sum_0^\infty \zeta(n) z^n$ has no (finite) singularity in T.

Corollary 2. If $f(z) = \sum_0^\infty a_n z^n$, analytic about the origin, has a (finite) singularity in T, then no entire function $\zeta(z) = \sum_0^\infty b_n z^n$ exists with $|b_n| \leqq M/n!$ such that $\zeta(n) = a_n$.

THEOREM 4. *Given the set of numbers a_n, $n = 0, 1, \cdots$. A necessary condition that $\zeta(z) = \sum_0^\infty b_n z^n$, entire, exist with $|b_n| \leqq M/n!$ for some M, such that $\zeta(n) = a_n$, is that $|a_n| \leqq Me^n$.*

Proof: We are given that $\zeta(z)$ exists. Therefore $|\zeta(0)| \leqq M$, $|\zeta(k)| \leqq M \sum_0^\infty (k^n/n!) = Me^k$. This proves the theorem since $a_k = \zeta(k)$.

Corollary. If $\zeta(z) = \sum_0^\infty b_n z^n$ is entire with $|b_n| \leqq M/n!$ and such that either $\zeta(2n) = 0$ or $\zeta(2n+1) = 0$, $n = 0, 1, \cdots$, then $F(z) = \sum \zeta(n) z^n$ is entire.

Proof: $F(z)$ is analytic in T. But also $F(z)$ is either an odd function or an even function, and is therefore analytic in T', the image of T in the

* In fact, the series converges in $|z| < 1/e$ at least.

imaginary axis. $T + T'$ gives every point of the z-plane except the origin. But by Theorem 4, $F(z)$ is analytic at $z = 0$. Hence $F(z)$ is entire.

We come now to Borel's theorem:

THEOREM. *If* $\zeta(z) = \sum_0^\infty b_n z^n$ *is an entire function of genus zero, then* $f(z) = \sum_0^\infty \zeta(n) z^n$ *is analytic everywhere (in the finite plane) save possibly at* $z = 1$.

Proof: We use the following property * of a function of genus zero:

$n! \mid b_n \mid c^n \to 0$ for every c however large (but fixed).

Now $f(z) = \sum_0^\infty b_n u_n(z) = \sum_0^\infty b_n P_n(z)/(1-z)^{n+1}$; and for $z \neq 1$,

$$\sum_0^\infty \left| b_n \frac{P_n(z)}{(1-z)^{n+1}} \right| \leqq \frac{M}{|1-z|} \sum_0^\infty \left(\frac{l}{c\,|1-z|}\right).\dagger$$

If we suppose c chosen so that $l/c\,|1-z| < 1$, we have convergence. Uniform convergence in any closed (finite) region not including $z = 1$ follows at once. This proves the theorem.

We remark that the possibility of $z = 1$ being a singular point is not illusory. Consider for example $\zeta(z) = z$. Here $\zeta(n) = n$, so that $f(z) = \sum_0^\infty nz^n = z/(1-z)^2$.

Related to the polynomials $P_n(z)$ is another set $Q_n(z)$ defined by $Q_0(z) = 1$, $zQ_n(z) = P_{n+1}(z)$, $n > 0$, which has some interesting properties. We state a few of these properties:

1) $Q_n(z)$ is a real polynomial of degree n, with positive integral coefficients.

2) $Q_n(z)$ satisfies the equation

(10) $\quad Q_n(z) = z(1-z)Q'_{n-1}(z) + (nz+1)Q_{n-1}(z)$, $n > 0$.

3) $Q_n(z)$ is a reciprocal polynomial. (That is, for every zero ξ there is a zero $1/\xi$.)

3′) $Q_{2n+1}(-1) = 0$.

4) The zeros of $Q_n(z)$ are real, simple, and (except for the zero $z = -1$ when n is odd) irrational; they separate the zeros of $Q_{n-1}(z)$; and they all lie in the interval $-\infty < z < 0$.

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* Due to Poincaré. Cf. Borel, *Fonctions Entières*, pp. 53-55.

† l is the larger of $|z|$ and 1.

Optics in Space of Constant Non-Vanishing Curvature.

By James Pierpont

1. *Introduction.* In a paper presented at the last meeting of the Society (Dec. 1926)* I showed that in such a space central optical collineation gives only an image congruent with the object. As is well known the theory of collineation enables one to establish a body of theorems of great importance in e-optics † and the fact that this theory is not available in E- and H-spaces would prove a great bar to our study of optics in such spaces if some other general method were not at hand. It is the purpose of this paper to outline a method which is partially successful, in the hope that others, more competent, may greatly improve thereon. In any case the method as here developed reaches results that can not be obtained by other means as far as known to the author.

The essential difference between this and the foregoing paper lies in this: In the former, light was regarded from the standpoint of rays, here it is regarded as a wave phenomenon.

I was led to take this view by a very remarkable memoir ‡ by C. S. Hastings "On Certain New Methods and Results in Optics" and this present paper must be regarded as an attempt to extend Hastings' methods with more or less success to space of constant non-vanishing curvature. I shall however for the sake of clearness speak only of H-space but it will be seen that the methods hold for E-space as well.

2. *Hyperbolic Geometry.* As one can define the metric of H-geometry in a variety of ways we will begin by recalling the particular form adopted here and then develop a few results needed in the optical part of this paper. Let x, y, z be three real variables which we regard as defining a point. Let $R > 0$ a constant

* "Optics in Hyperbolic Space." The method there used can be applied without trouble to elliptic space.

† For *e-*, *H-*, *E-* read euclidean, hyperbolic, elliptic.

‡ *Memoirs of the National Academy of Sciences*, Vol. 6 (1893), p. 37. Reprinted with slight changes in his book *Light*, Yale Bicentennial Publications. Scribners, New York, 1901.

$$r^2 = x^2 + y^2 + z^2, \quad \lambda = 4R^2 - r^2 > 0, \quad ds^2 = dx^2 + dy^2 + dz^2.$$

The metric is then defined by

$$d\sigma = 4R^2 ds/\lambda. \tag{1}$$

This is Riemann's definition. To form a concrete picture of abstract H-geometry we may regard x, y, z as ordinary cartesian coordinates of a point. Then the points of H-space are represented in the model by the points lying within the e-sphere $\lambda = 0$. For points near the origin 0 we see $d\sigma = ds$ nearly, a fact we shall employ.

H-planes are represented in the model by e-spheres cutting $\lambda = 0$ orthogonally. The intersection of two H-planes is an H-straight. In particular H-planes through O are in the model also e-planes. H-spheres are e-spheres in the model whose centers are not those of the H-spheres unless at the origin. Angles in H-geometry are the same as the corresponding angles in the model. The model is thus conformal, a fact we largely employ in the sequel. Bodies in H-space may be moved about freely such that the distance $d\sigma$ between adjacent points remains unaltered. This fact is most useful as it enables us to move a certain point to the origin O when H-planes and straights through it become e-planes and straights while angles remain unaltered. From (1) we find that the H-length σ of a straight segment OP, $P = (xyz)$ is given by

$$s = 2R \tanh (\sigma/2R). \tag{2}$$

If s receives the increment ds

$$ds = \operatorname{sech}^2 (\sigma/2R) \cdot d\sigma. \tag{3}$$

The H-length of an arc ϕ of an H-circle K, center A, of radius ρ is

$$\sigma = R\phi \sinh (\rho/R). \tag{4}$$

Its curvature

$$\Gamma = d\phi/d\sigma = 1/R \sinh (\rho/R). \tag{5}$$

K regarded as an e-circle has its center A_1 on the straight OA, its radius call r. Let this straight cut K in B and C. Let

$$\begin{aligned} OA_1 &= a, \quad & OB &= b, \quad & OC &= c, \text{ in e-measure,} \\ &= \alpha, & &= \beta, & &= \gamma, \text{ in H-measure.} \end{aligned}$$

Then by (2)

$$2r = c - b = 2R\{\tanh(\gamma/2R) - \tanh(\beta/2R)\}, \qquad \gamma - \beta = 2\rho.$$

Hence
$$r = \frac{R \sinh(\rho/R)}{\cosh(\gamma/2R)\cosh(\beta/2R)}.$$

If K passes through O, $\beta = 0$, $\gamma = 2R$, whence

(6)
$$r = R\tanh(\rho/R).$$

In the following we shall deal with H-spherical light waves and spherical lenses which pass through or near O, the relation (6) is thus true if we neglect small quantities of order greater than 1 which we do throughout and which is also done in this kind of work in e-optics.

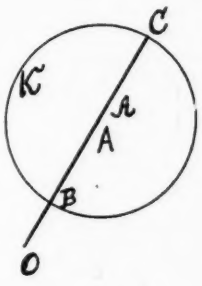

FIG. 1.

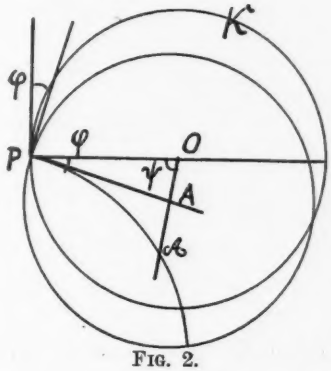

FIG. 2.

In figure 2 an H-circle K of radius $\rho(r)$* and center O is turned through the angle ϕ at P. Let the new circle have e-radius r'. We need to know how much r' differs from r where ϕ is small.

Let $POA = \psi$, $\psi' = 90 - \psi$, $PA = s$. Then
$$\tan\psi' = \cosh(\rho/R)\tan(\phi/2)$$

$$s = \frac{r\cos\psi'}{\cos(\phi - \psi')} = \frac{r}{\cos\phi + \sin\phi\tan\psi'}.$$

Hence
$$s = \frac{r}{\cos\phi + \sin\phi\tan(\phi/2)\cosh(\rho/R)}.$$

If ϕ is small

(7)
$$s = \frac{r}{1 + (\phi^2/2)\cosh(\rho/R)}.$$

* Means, radius is ρ in H-measure and r in e-measure.

Hence for small enough ϕ we may take $s = r$.

In figure 3, K is an H-circle whose center is O and radius $\rho(r)$. We turn K about P through an angle ϕ getting K' whose e-center Ω has coordinates $(-\alpha, -\beta)$. The straight OQ makes the small angle θ with OP and cuts K in Q_1; the segment QQ_1, has length $\delta(d)$. The H-length of PQ is σ. We wish to establish the important relation

$$\delta = \sigma \tan \phi, \tag{8}$$

neglecting small quantities of order > 1.

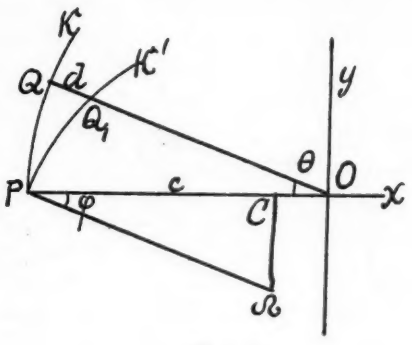

FIG. 3.

To this end regard K' as an e-circle obtained by rotating K through the angle ϕ. Then $PQ = P\Omega = r$. Let $PC = c$. The equation of K' is

$$(x + \alpha)^2 + (y + \beta)^2 = r^2.$$

The equation of OQ is $y = -x \tan \theta = -x\theta$.

Thus the x coordinate of Q_1 in numerical value is

$$r_1 = r - \beta\theta(r/c) = r - r\theta \tan \phi.$$

The projection of QQ_1 on the x-axis is $d \cos \theta = r - r_1 = r\theta \tan \phi$. Thus neglecting θ^2, etc.,

$$d = r\theta \tan \phi. \tag{9}$$

We obtain (8) from (9) as follows. Suppose $P\Omega = r$, nearly, as it certainly is if ϕ is small.

By (3) $\qquad d = \delta \operatorname{sech}^2 (\rho/2R).$

By (4) $\qquad \theta = \sigma/R \sinh (\rho/R),$ $\qquad$ by (2) $\quad r = 2R \tanh (\rho/2R).$

These in (9) give (8) at once.

3. *Relations between Curvatures.* In figure 4, LML' is a cross-section of a spherical lense of H-curvature Λ. A spherical wave meets the lense and for small angles of incidence, is bent into another spherical wave. Let Γ, Γ' be the H-curvatures of the wave fronts LWL', $LW'L'$. Thus while the incident wave front travels from M to W the refracted wave travels from M to W'. We have then the relation

(10) $$MW' = nMW, \quad n \text{ index of refraction.}$$

But $MW' = MN - NW'$, $MW = MN - NW$. Hence

(11) $$MN - NW' = n(MN - NW).$$

We may regard these H-spheres or rather H-circles as e-circles whose e-curva-

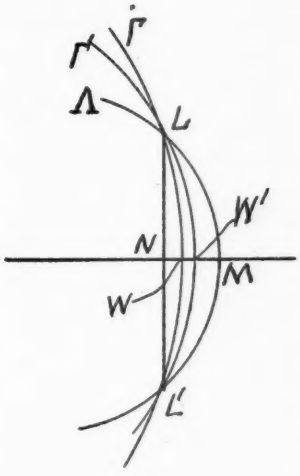

FIG. 4.

tures are L, C, C' respectively. Then setting $LN = p$ in e-measure

$$NM = \tfrac{1}{2}p^2L, \quad NW = \tfrac{1}{2}p^2C, \quad NW' = \tfrac{1}{2}p^2C'.$$

These in (11) give

(12) $$L - C' = n(L - C) \text{ or } C' = mL + nC, \quad (m = 1 - n).$$

From (12) we have denoting the radii of these circles by r', l, r respectively,

$$1/r' = m/l + n/r.$$

Let us now pass back to H-space. If we take our origin O at or near M we have using (6)

$$\coth(\rho'/R) = m\coth(\lambda/R) + n\coth(\rho/R) \tag{13}$$

where ρ', λ, ρ are the radii of the above circles regarded as H-circles. By giving C, C', L in (12) appropriate signs the relation (12) is valid for all cases. For example, a convex lense with n such that C, C' have opposite curvatures we have $C' = mL - nC$, whence

$$\coth(\rho'/R) = m\coth(\lambda/R) - n\coth(\rho/R). \tag{14}$$

Let us compare this result with that given in my December paper mentioned above for a particular case:

$$R = 1000, \quad a = 600, \quad r = 10, \quad n = 2/3, \quad \phi = 30'.$$

We had

$$\cos\chi = \cos(\alpha-\beta)\cos\theta + \sin(\alpha-\beta)\sin\theta\cosh(\sigma/R)$$

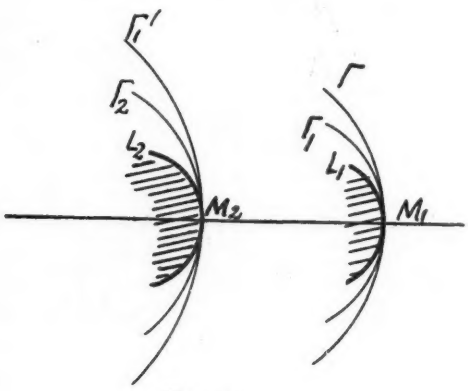

FIG. 5.

or setting $\cosh(\sigma/R) = 1 + C'$, and noting $\alpha - \beta - \theta = \phi - \beta$

$$\cos\chi = \cos(\phi-\beta) + C'\sin\theta\cdot\sin(\alpha-\beta) = \cos(\phi-\beta) + E,$$

$$\sinh(\eta/R) = \frac{\sin(\alpha-\beta)}{\sin\chi}\cdot\sinh(\sigma/R).$$

We find:

$$\theta = .0001479073, \quad \alpha = \theta + \phi = .008874553, \quad \beta = .0059163688,$$
$$\delta = \phi - \beta = .002810277, \quad \alpha - \beta = .002958184,$$
$$\sigma = 608{,}06, \quad C' = .1906, \quad E = 8{,}3.10^{-8},$$
$$\chi = .002780641, \quad \eta = 642.37, \quad \rho' = \eta - \sigma = 34{,}3.$$

If we use (14) we get $\rho' = 34{,}3$ in perfect agreement.

In figure 5 the spherical wave front of curvature Γ strikes the interface of a lense L_1 and is bent into the spherical wave front of radius ρ_1, and curvature Γ_1. It then strikes the interface L_2. If η_1 is the H-distance M_1M_2 the radius of the wave front at L_2 is $\rho_1 - \eta_1$ hence its curvature is

$$\Gamma_1' = \frac{1}{R \sinh [(\rho_1 - \eta_1)/R]} = \frac{1}{\sinh (\rho_1/R) - (\eta_1/R) \cosh (\rho_1/R)},$$

since η_1/R is small. Hence

$$\Gamma_1' = \frac{\Gamma_1}{1 - \eta_1\Gamma_1 \cosh (\rho_1/R)} = g_2\Gamma_1; \quad 1/g_2 = 1 - \eta_1\Gamma_1 \cos (\rho_1/R). \tag{15}$$

The lense interface L_2 gives Γ_1' the curvature Γ_2 etc.

To get relations that are easily handled we may regard these wave fronts as e-spheres whose curvatures are C, C_1, C_1', C_2. Let $M_1M_2 = h_1$ in e-measure. The radius of the first refracted wave front at M_1 is $1/C_1$, at M_2 its radius is $(1/C_1) - h_1 = C_1/(1 - h_1C_1)$. Thus its curvature at M_2 is

$$C_1' = \mu_2 C_1, \quad \mu_2 = 1/(1 - h_1C_1).$$

We have then analogous to (12) for a set of lenses

$$\begin{aligned} C_1 &= m_1L_1 + n_1\mu_1C \\ C_2 &= m_2L_2 + n_2\mu_2C_1 \\ C_3 &= m_3L_3 + n_3\mu_3C_2, \text{ etc.} \end{aligned} \tag{16}$$

Here $\mu_1 = 1$ is set in for symmetry, n_k is the index at the kth interface L_k $m_k = 1 - n_k$, $1/\mu_k = 1 - h_{k-1}C_{k-1}$, h_k is the e-distance between the faces $L_{k-1}L_k$, L_k is the e-curvature of the length L_k. It is understood that the curvatures are taken with their proper sign. The relations (16) are the (a) of Hastings' paper. They are fundamental in his theory.

As an example, let us consider a thin convex lense in air. The lense being thin we may take $h_1 = 0$. Then at the first interface L_1,

$$C_1 = n_1C - m_1L_1.$$

At the second interface L_2,

$$C_2 = m_2L_2 - n_2C_1.$$

Set $n_1 = n$, then $n_2 = 1/n$, $m_2 = (n-1)/n$, hence

$$C_1 + C_2 = (n-1)/n(L_1 + L_2), \; 1/r_1 + 1/r_2 = [(n-1)/n](1/l_1 + 1/l_2)$$

or by (6)

$$\coth (\rho_1/R) + \coth (\rho_2/R) = [(n-1)/n][\coth (\lambda_1/R) + \coth (\lambda_2/R)]. \tag{17}$$

Here λ_1, λ_2 are the radii of the two faces of the lense; if ρ_1 is the distance of the source of light, ρ_2 is the distance of the image, all in H-measure. The extension of this result to a system of thin lenses in contact is obvious. It is hard to see how such results can be obtained by the trigonometric calculations of rays as in my first paper.

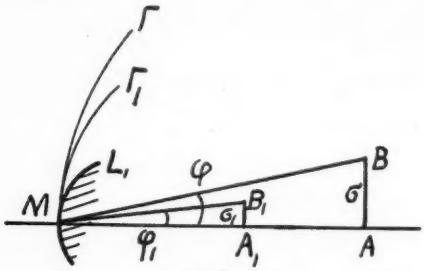

FIG. 6.

4. *Magnification. The Sine Law.* In figure 6, the small object AB of length σ becomes A_1B_1 of length σ_1 after refraction. The wave front of radius ρ and curvature Γ, has radius ρ_1 and curvature Γ_1 after refraction. We have from the right H-triangle

$$\sinh(\sigma/R) = \sin\phi \sinh(\rho/R) \quad \text{or} \quad \sigma/R = \phi \sinh(\rho/R).$$

Similarly $\sigma_1/R = \phi_1 \sinh(\rho_1/R)$. Hence

$$\sigma_1/\sigma = \phi_1/\phi \cdot \sinh(\rho_1/R)/\sinh(\rho/R).$$

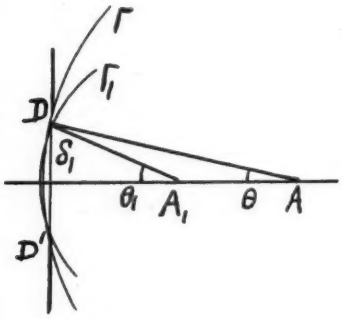

FIG. 7.

Now $\theta_1/\theta = n$ the index of refraction, hence by (15)

$$\sigma_1/\sigma = n_1(\Gamma/\Gamma_1) = n_1 g_1(\Gamma/\Gamma_1), \quad g_1 = 1,$$

g_1 being introduced for symmetry. After a 2nd refraction we have similarly

$$\sigma_2/\sigma_1 = n_2 g_2(\Gamma_1/\Gamma_2),$$

and after p refractions

$$\sigma_p/\sigma_{p-1} = n_p g_p(\Gamma_{p-1}/\Gamma_p).$$

Here $g_2, g_3, \cdots$ are as in (15). Multiplying these, gives as magnification

$$M = \sigma_p/\sigma = GN(\Gamma/\Gamma_p) \qquad (G = g_1 g_2 \cdots g_p,\ N = n_1 n_2 \cdots n_p). \tag{18}$$

For a thin lense system we may take $G = 1$. Suppose we have a small diaphragm $D\,D'$ of diameter $2\delta_1$ at the first lense as in figure 7. Then if $A_1 D = \rho_1$, $A\,D = \rho$.

$$\sin\theta = \sinh(\delta_1/R)/\sinh(\rho/R) = \delta_1\Gamma$$

$$\sin\theta_1 = \delta_1\Gamma_1.$$

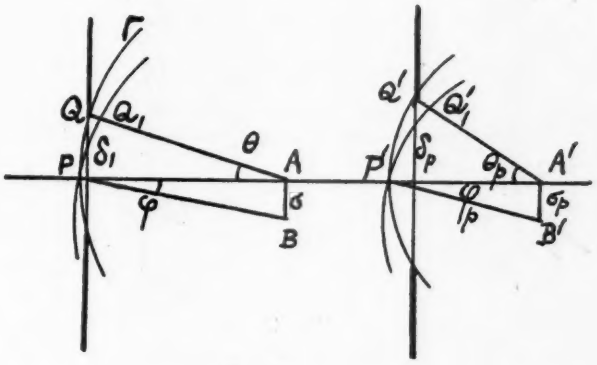

FIG. 8.

Hence

$$\sin\theta_1/\sin\theta = \Gamma/\Gamma_1 = g_1(\Gamma/\Gamma_1), \quad g_1 = 1.$$

Similarly diaphragms $\delta_2 \cdots \delta_p$ at the other interfaces give

$$\sin\theta_1/\sin\theta_2 = g_2(\Gamma_1/\Gamma_2), \cdots, \quad \sin\theta_{p-1}/\sin\theta_p = g_p(\Gamma_{p-1}/\Gamma_p).$$

Multiplying these, gives

$$\sin\theta/\sin\theta_p = G(\Gamma/\Gamma_p).$$

This with (18) gives

$$\sigma_p/\sigma = N \sin\theta/\sin\theta_p = N\theta/\theta_p, \quad \theta,\ \theta_p \text{ small.} \tag{19}$$

This is the celebrated sine law of Lagrange.

Hastings by an ingenious consideration shows that (19) holds when θ, θ_p are not small, provided the incident and emergent wave fronts are spherical, i. e., the instrument is free from spherical aberration. In figure 8, $P\,Q$ is the incident wave front issuing from A and $P\,Q_1$ that issuing from B the other extremity of a small object σ.

In the other half of the figure $P'Q'$, $P'Q'_1$ have a similar meaning. The incident wave front is limited by a diaphragm of semi-diameter δ_1. The diaphragm δ_p is the optical image of δ_1. Then Q_1, Q'_1 are corresponding points and hence setting $QQ_1 = \epsilon$, $Q'Q'_1 = \epsilon_p$ in H-measure,

$$\epsilon_p = N\epsilon \tag{20}$$

since the velocity of light in the last medium is N times that in the first. To prove (20) let us refer to figure 3 which we suppose first is euclidean. Let $e = QQ_1$ in e-measure. The equation of the e-circle K' is as before, while

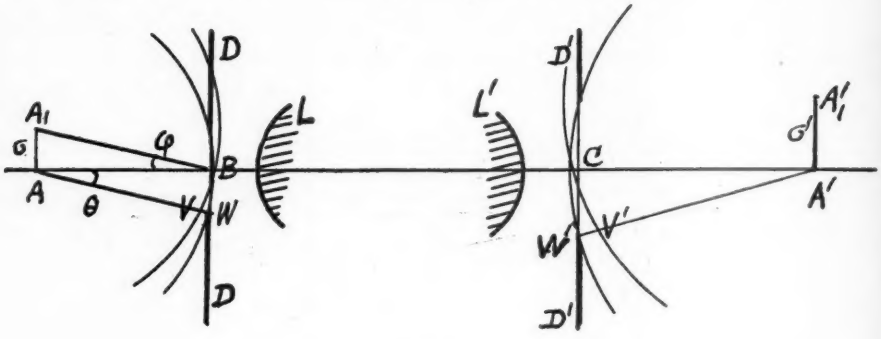

FIG 9.

the equation of the straight OQ is $y = -x \tan\theta$ where now θ is not small. The x coordinate of Q_1 is now in numerical value $r_1 = r\cos - \frac{1}{2}r\phi \sin 2\theta$. The projection of QQ_1 on the x axis is

$$e\cos\theta = r\cos\theta - r_1 = \tfrac{1}{2}r\phi \sin 2\theta.$$

Hence $$e = r\phi \sin\theta.$$

Using as before (2), (3), (4), $$\epsilon = \sigma \sin\theta.$$

Similarly $$\epsilon_p = \sigma_p \sin\theta_p.$$

These with (20) give

$$\epsilon_p/\epsilon = N = \sigma_p \sin\theta_p/\sigma \sin\theta.$$

Hence

$$\sigma_p/\sigma = N \sin\theta/\sin\theta_p, \tag{21}$$

which is (19) with this difference that θ, θ_p are not restricted to small values.

5. *Telescopes, Microscopes.* In figure 9, L and L' are object glass and ocular of a telescope or microscope. In front of L is a small diaphragm of semi-diameter δ, behind L' is another of semi-diameter δ', so placed that D' is the optical image of D. The extremities of $A\,A_1$ of an object of length σ send out light waves which at B are inclined at an angle ϕ. We are now under the condition of figure 3. Thus by (8)

$$VW = BW \tan\phi = \delta \tan\phi.$$

Similarly on the right we have

$$V'W' = B'W' \tan\phi' = \delta' \tan\phi'.$$

Now N being the ratio of the velocity of light at the two ends $V'W' = N \cdot VW$. Thus

$$\tan\phi'/\tan\phi = N(\delta/\delta'). \tag{22}$$

Let us now suppose figure 9 represents a telescope. Let $AB = \alpha$, $BC = \tau$, $\alpha + \tau = \beta$. Let σ subtend an angle ϕ at B, an angle ψ at C, and an angle ϕ' as seen through the instrument. We define the magnification afforded by the instrument by

$$M = \phi'/\psi. \tag{23}$$

Now from figure 9

$$\sigma = R\phi \sinh(\alpha/R) = R\psi \sinh(\beta/R).$$

Thus using (22)

$$M = \frac{\phi'}{\phi} \cdot \frac{\phi}{\psi} = N \frac{\delta}{\delta'} \frac{\sinh(\beta/R)}{\sinh(\alpha/R)}. \tag{24}$$

For a telescope $N = 1$ also τ/R is small, and hence we can take $\alpha = \beta$. Thus

$$M = \delta/\delta'.$$

Let us take the diaphragm δ to be the edge of the object glass, and δ' its ocular image. Hence Ramsden's rule

$$M = \frac{\text{diameter of objective}}{\text{diameter of its ocular image}} \tag{25}$$

holds for E- and H-spaces.

For a microscope in e-space the magnification is taken as

$$M = \frac{\text{size as seen through instrument}}{\text{size as seen at distance 10 inches}}$$

As we know nothing about the eyes of beings in other spaces than our own, let us suppose ω is the distance of easy vision and Σ is the apparent size of the object as seen through the instrument, coming from the distance ω. We may take as magnification

$$M = \frac{\tanh(\Sigma/R)}{\tanh(\sigma/R)} \tag{26}$$

But $\tanh(\Sigma/R) = \sinh(\omega/R)\tan\phi'$, $\tanh(\sigma/R) = \sinh(\omega/R)\tan\psi$.

Thus

$$M = \frac{\tan\phi'}{\tan\psi} = \frac{\tan\phi'}{\tan\phi}\cdot\frac{\tan\phi}{\tan\psi} = N\cdot\frac{\delta}{\delta'}\cdot\frac{\tan\phi}{\tan\psi}$$

Now

$$\frac{\tan\phi}{\tan\psi} = \frac{\sigma}{\alpha}\cdot\frac{\omega}{\sigma} \quad \text{, approximately.}$$

Hence

$$M = N\,\frac{\delta}{\delta'}\cdot\frac{\omega}{\alpha} \quad \text{, for a microscope.} \tag{27}$$

There are other results which may be obtained from Hastings' memoir under certain restrictions, we will however stop at this point.

On an Imprimitive Group of Order 5184.*

By J. R. Musselman.

In the study of collineation groups certain types of groups which, in the plane, have a fixed triangle and, in space, have a fixed tetrahedron are classed as imprimitive. The existence of such groups is clear and their group structure is easily ascertained. Thus, from the group standpoint, the real problem consists of the determination of the primitive collineation groups. As a result the configurations determined by primitive groups have been studied rather completely and those determined by imprimitive groups have been rather neglected. The object of this paper is to discuss a configuration of 108 lines, 216 points and 216 planes with which is associated an imprimitive group of order 5184. Connected with this configuration are eight cubic surfaces, eighteen quadric surfaces, and fifty-four tetrahedra, whose geometrical properties are described, as well as the group of order 5184 which leaves the configuration invariant.

1. *The Bilinear Invariant of Two Cubics.* Two binary cubics

$$a_0x_0^3 + 3a_1x_0^2x_1 + 3a_2x_0x_1^2 + a_3x_1^3$$

and

$$a_0'x_0^3 + 3a_1'x_0^2x_1 + 3a_2'x_0x_1^2 + a_3x_1^3$$

have the bilinear invariant

$$a_0a_3' - 3a_1a_2' + 3a_2a_1' - a_3a_0'. \tag{1}$$

This invariant can be expressed in terms of the symmetric functions of the roots of the above two cubics as

$$3s_3' - s_2's_1 + s_1's_2 - 3s_3 \tag{2}$$

where

$$s_3' = a_3'/a_0', \qquad s_3 = a_3/a_0, \text{ etc.}$$

If we designate the roots of the two cubics as α_0, α_1, α_2 and β_0, β_1, β_2 respectively it is well known that the invariant (1) can be expressed in terms of the differences of the roots as

* Presented to the American Mathematical Society, Dec. 29, 1924 and May 7, 1927.

$$\sum^{6} (\alpha_0 - \beta_0)(\alpha_1 - \beta_1)(\alpha_2 - \beta_2).$$

It is unnecessary, however, to use all six terms of the above sum of differences to express the function (2). For it is evident that the function

$$\gamma \equiv (\alpha_0 - \beta_0)(\alpha_1 - \beta_1)(\alpha_2 - \beta_2) + (\alpha_0 - \beta_1)(\alpha_1 - \beta_2)(\alpha_2 - \beta_0) + (\alpha_0 - \beta_2)(\alpha_0 - \beta_0)(\alpha_2 - \beta_1)$$

when expressed in terms of the symmetric functions of the α_i and β_i $(i = 0, 1, 2)$ is precisely (2) itself.

This fact suggests two questions—given two n-ics how many terms of differences of their roots it is necessary to use to form the bilinear invariant when expressed in symmetric functions of the roots of the n-ics; and, secondly, given one term of differences, how can we select the others that must go with it to form the required function?

A little investigation will show that in the general case of two binary n-ics the number of terms of differences of the roots necessary to form the function (2) is the least common multiple of the binomial coefficients of the n-ic. For the case $n = 2$, it is necessary to use both terms; for the case $n = 3$, the number of terms of differences of the roots necessary to form (2) is only three.

The answer to the second question will be discussed elsewhere for the general n-ics. For the case $n = 3$, given any term as $(\alpha_0 - \beta_0)(\alpha_1 - \beta_1)(\alpha_2 - \beta_2)$ to obtain the other terms which together with it produce the function (2) we must keep the α_i fixed and permute the β_i cyclically. Also if we had kept the β_i fixed and permuted the α_i cyclically we likewise should have obtained the function γ. *Hence given any term of differences, to obtain the two terms to go with it, carry out on the given term, the even permutations of the symmetric $G_{3!}$ of either set of the roots.*

The three remaining terms of differences

$$\gamma' \equiv (\alpha_0 - \beta_0)(\alpha_1 - \beta_2)(\alpha_2 - \beta_1) + (\alpha_0 - \beta_2)(\alpha_1 - \beta_1)(\alpha_2 - \beta_0) + (\alpha_0 - \beta_1)(\alpha_1 - \beta_0)(\alpha_2 - \beta_2)$$

will likewise produce the function (2). It is evident that the two functions γ and γ' are formally multiply isomorphic with the $G_{3!}$ on either set of roots; since both γ and γ' are unaltered by $\{1, (\alpha_0\alpha_1\alpha_2), (\alpha_0\alpha_2\alpha_1)\}$; and are interchanged by the permutations $(\alpha_0\alpha_1)$, $(\alpha_0\alpha_2)$, $(\alpha_1\alpha_2)$. We have here an illustration of a $3 - 1$ isomorphism with the symmetric $G_{3!}$

2. *The Cubic Surface and a G_{36}.* The purpose of this paper is to study the equation $\gamma = 0$ or $\gamma' = 0$. Let us write $\gamma = 0$ in the more familiar form

$$(x_1 - x_4)(x_2 - x_5)(x_3 - x_6) + (x_1 - x_5)(x_2 - x_6)(x_3 - x_4) + (x_1 - x_6)(x_2 - x_4)(x_3 - x_5) = 0.$$

In projective coordinates, this represents a cubic manifold in space of five dimensions, with in evidence, 27 planes on it, such as $x_1 - x_4 = x_2 - x_6 = x_3 - x_5 = 0$, which planes all pass through the point (1, 1, 1, 1, 1, 1). If we take a section of the manifold by the S_3 $x_1 + x_2 + x_3 = x_4 + x_5 + x_6 = 0$ we obtain thereby a cubic surface in S_3 which is invariant at least under a G_{72}, generated by the $G_{3!}$ on x_1, x_2, x_3; the $G_{3!}$ on x_4, x_5, x_6; and the G_2 which interchanges x_1, x_4; x_2, x_5; and x_3, x_6.

The equation of the cubic surface in S_3 will now take the form

$$(3) \qquad \begin{cases} x_1x_2x_3 = x_4x_5x_6 \\ x_1 + x_2 + x_3 = x_4 + x_5 + x_6 = 0. \end{cases}$$

For our purpose it will be sufficient to find the fixed points, fixed lines and fixed planes of the G_{36}, generated by the two symmetric $G_{3!}$'s. In this G_{36} are 15 collineations of the type $(\beta_1\beta_2)$ which leave invariant 6 points and 9 lines, such that 3 lines are on each point. They also leave invariant 6 planes and 9 other lines, such that 3 lines are in each plane. The 6 points and the 6 planes are on two other lines. The coordinates of the 6 points are as follows:

p_0: (0, 1, −1, 0, 0, 0) $\qquad$ q_0: (0, 0, 0, 0, 1, −1)

p_1: (1, 0, −1, 0, 0, 0) $\qquad$ q_1: (0, 0, 0, 1, 0, −1)

p_2: (1, −1, 0, 0, 0, 0) $\qquad$ q_2: (0, 0, 0, 1, −1, 0)

To write the equations of the lines and planes compactly, let us write x_0, x_1, x_2 for x_1, x_2, x_3 and y_0, y_1, y_2 for x_4, x_5, x_6 and take subscripts modulo 3. Then the 6 planes can be written

P_i: $x_{i+1} - x_{i-1} = 0 \qquad x_0 + x_1 + x_2 = y_0 + y_1 + y_2 = 0.$

Q_i: $y_{i+1} - y_{i-1} = 0 \qquad x_0 + x_1 + x_2 = y_0 + y_1 + y_2 = 0.$

The 9 lines on the 6 points can be written

$$(4) \quad \overline{p_iq_k}^4: \; x_{i+1} + x_{i-1} = y_{k+1} + y_{k-1} = 0 \quad x_0 + x_1 + x_2 = y_0 + y_1 + y_2 = 0.$$

The 9 lines on the 6 planes can be written

$$(5)\quad \overline{P_iQ_k}\colon\ x_{i+1}-x_{i-1}=y_{i+1}-y_{i-1}=0,\ x_0+x_1+x_2=y_0+y_1+y_2=0.$$

We shall denote the first set of 9 lines by a, the second by a'.

In the G_{36} are 4 collineations of the type $(\beta_1\beta_2\beta_3)$ which leave invariant two lines and four points.

$$L_1\colon\ x_0-x_1=x_1-x_2=x_2-x_0=0 \quad x_0+x_1+x_2=y_0+y_1+y_2=0.$$

$$L_2\colon\ y_0-y_1=y_1-y_2=y_2-y_0=0 \quad x_0+x_1+x_2=y_0+y_1+y_2=0.$$

$$r\colon\ (0,0,0,\omega,\omega^2,1) \qquad t\colon\ (\omega,\omega^2,1,0,0,0)$$

$$s\colon\ (0,0,0,\omega^2,\omega,1) \qquad u\colon\ (\omega^2,\omega,1,0,0,0).$$

On L_1 are the points q_i, the planes P_i and the points r and s are the Hessian points of the set q_i; similarly on L_2 are the points p_i, the planes Q_i and the points t and u are the Hessian points of the set p_i.

In the G_{36} are 4 collineations of the type $(\beta_1\beta_2\beta_3)(\alpha_1\alpha_2\alpha_3)$ which have as fixed lines the four lines $\overline{rt}$, $\overline{ru}$, $\overline{st}$ and $\overline{su}$. The remaining 12 collineations of the type $(\beta_1\beta_2\beta_3)(\alpha_1\alpha_2)$ have as fixed points, in addition to the ten already named, six others lying on L_1 and L_2; which points are the cubicovariant points of the sets of p_i and q_i respectively. Moreover, the 9 lines a are the joins of the p_i and q_i; the 9 lines a' are the joins of the cubicovariant sets of p_i and q_i.

3. *The 27 Lines of the Surface.* It is evident that the 9 lines a are contained in the surface (3). The equations of the remaining 18 lines of the surface can be written down without difficulty. There is a second set of 9 lines which we shall call b whose equations are

$$(6)\quad x_i-\omega^k y_i=x_{i+1}-\omega^k y_{i-1}=x_{i-1}-\omega^k y_{i+1}=0 \quad \Sigma x=\Sigma y=0$$
$$(i,k=0,1,2);\quad \omega^3=1.$$

These 9 lines meet by threes on 6 points which lie on two lines

$$L_3\colon\ x_0-\omega x_1=y_0-\omega y_2=0 \qquad \Sigma x=\Sigma y=0$$

$$L_4\colon\ x_0-\omega x_2=y_1-\omega y_2=0 \qquad \Sigma x=\Sigma y=0.$$

On L_3 are the points

$$(1,\omega^2,\omega,\omega,\omega^2,1);\ (\omega,1,\omega^2,\omega,\omega^2,1);\ (\omega^2,\omega,1,\omega,\omega^2,1).$$

On L_4 are the points

$$(1, \omega, \omega^2, \omega^2, \omega, 1); \quad (\omega, \omega^2, 1, \omega^2, \omega, 1); \quad (\omega^2, 1, \omega, \omega^2, \omega, 1).$$

Moreover the points r, and u are the Hessian points of the set on L_3; similarly s and t for the set on L_4.

The third set of 9 lines which we shall call c have for their equations

$$(7) \qquad x_0 - \omega^k y_{0+i} = x_1 - \omega^k y_{1+i} = x_2 - \omega^k y_{2+i} = 0 \qquad \Sigma x = \Sigma y = 0$$
$$(i, k = 0, 1, 2); \quad \omega^3 = 1.$$

These 9 lines meet by threes on 6 points which lie on two lines

$$L_5: \; x_0 - \omega x_2 = y_0 - \omega y_2 = 0 \qquad \Sigma x = \Sigma y = 0$$
$$L_6: \; x_0 - \omega x_1 = y_1 - \omega y_2 = 0 \qquad \Sigma x = \Sigma y = 0.$$

On L_5 are the points

$$(1, \omega, \omega^2, \omega, \omega^2, 1); \quad (\omega, \omega^2, 1, \omega, \omega^2, 1); \quad (\omega^2, 1, \omega, \omega, \omega^2, 1).$$

On L_6 are the points

$$(1, \omega^2, \omega, \omega^2, \omega, 1); \quad (\omega, 1, \omega^2, \omega^2, \omega, 1); \quad (\omega^2, \omega, 1, \omega^2, \omega, 1).$$

Moreover the points r and t are the Hessian points of the set on L_5; similarly s and u for the set on L_6.

Now the lines L_3, L_4, L_5 and L_6 together with L_1 and L_2 are the edges of a tetrahedron of which the points r, s, t, u are the vertices. On each edge is a set of three points of which the vertices on that edge are the Hessian points. The 27 lines of the surface (3) are the joins of the sets of three points on the opposite edges. Since the 27 lines of the surface meet by threes in a plane in 18 points, this surface must be an Eckardt surface.*

On the edges L_1 and L_2 of the tetrahedron we have also the cubicovariant sets. Complete the configuration by finding the cubicovariant points of the sets on L_3, L_4, L_5 and L_6. Join the cubicovariant sets on L_3 and L_4 by lines, also those on L_5 and L_6. We obtain in this way two sets of 9 lines each which we shall denote by b' and c' respectively. Their equations are

$$(8) \qquad b': \; x_i + \omega^k y_i = x_{i+1} + \omega^k y_{i-1} = x_{i-1} + \omega^k y_{i+1} = 0 \qquad \Sigma x = \Sigma y = 0$$
$$c': \; x_0 + \omega^k y_{0+i} = x_1 + \omega^k y_{1+i} = x_2 + \omega^k y_{2+i} = 0 \qquad \Sigma x = \Sigma y = 0$$
$$(i, k = 0, 1, 2); \quad \omega^3 = 1.$$

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* Eckardt, *Mathematische Annalen*, Vol. 10 (1876), p. 265.

We thus have 54 lines meeting by threes in a plane at 36 points on the edges of a tetrahedron. Any line is also met by two others not in a plane, six times; the points of intersection falling into two sets of three each, of which the two points on the edge of the tetrahedron are their Hessian points. On each line lie 6 points and also 6 planes giving altogether a set of $54 \cdot 6/3 = 108$ points and 108 planes. On any given plane lie 15 points and on any point are 15 planes.

From the 54 lines we can select 27 in four different ways, which 27 lines lie on a surface of the Eckardt type. Thus the lines

a, b, c are on $x_0x_1x_2 = y_0y_1y_2$

a, b', c' are on $x_0x_1x_2 = -y_0y_1y_2$

a', b, c' are on $(x_0 - x_1)(x_1 - x_2)(x_2 - x_0) = -(y_0 - y_1)(y_1 - y_2)(y_2 - y_0)$

a', b', c are on $(x_0 - x_1)(x_1 - x_2)(x_2 - x_0) = (y_0 - y_1)(y_1 - y_2)(y_2 - y_0)$.

4. *The Configuration.* Let us start with a tetrahedron and determine the configuration from it. As the surface contained no constants, we have just 15 constants at our disposal; we shall use 12 of them by selecting the reference tetrahedron as our base. Select on any two opposite edges a set of three points such that the vertices are their Hessian points. This leaves one constant at our disposal and we shall select a similar set of three points on some other edge say L_3. The configuration must now be determined from the above conditions.

We have on L_1 points p_0: $(1, 1, 0, 0)$; p_1: $(\omega, 1, 0, 0)$; p_2: $(\omega^2, 1, 0, 0)$;

and on L_2 points q_0: $(0, 0, 1, 1)$; q_1: $(0, 0, \omega, 1)$; q_2: $(0, 0, \omega^2, 1)$.

The equations of the 9 lines joining the points on L_1 to those on L_2 are

$$(10) \qquad a_{ik} = \overline{p_iq_k}: \quad x_1 - \omega^i x_2 = x_3 - \omega^k x_4 = 0 \qquad (i, k = 0, 1, 2).$$

The cubicovariant set of points on L_1 are

p'_0: $(-1, 1, 0, 0)$; p'_1: $(-\omega, 1, 0, 0)$; p'_2: $(-\omega^2, 1, 0, 0)$

and the set on L_2 is

q'_0: $(0, 0, -1, 1)$; q'_1: $(0, 0, -\omega, 1)$; q'_2: $(0, 0, -\omega^2, 1)$.

The equations of the 9 lines joining these points are

$$(11) \qquad a'_{ik} = \overline{p'_i q'_k}: \quad x_1 + \omega^i x_2 = x_3 + \omega^k x_4 = 0 \qquad (i, k = 0, 1, 2).$$

The set of points on L_3 are $(1, 0, 0, 1)$; $(\omega, 0, 0, 1)$; $(\omega^2, 0, 0, 1)$.

The pencil of quadrics having L_1 and L_2 as one set of generators and L_3 and L_4 as opposite generators can be written

$$x_1 x_3 = \lambda\, x_2 x_4.$$

The three quadrics of this pencil which contain as generators a_{00}, a_{01}, a_{02} are cut on L_4 by the opposite generator through $(1, 0, 0, 1)$ on L_3 in the points $(0, 1, 1, 0)$; $(0, \omega, 1, 0,)$; $(0, \omega^2, 1, 0)$ respectively. The joins of the sets on L_3 and L_4 give the 9 lines

$$(12) \qquad c_{ik}: \quad x_1 - \omega^i x_4 = x_2 - \omega^k x_3 = 0 \qquad (i, k = 0, 1, 2).$$

If we join the cubicovariant sets of the above points on L_3 and L_4 we obtain the 9 lines

$$(13) \qquad c'_{ik}: \quad x_1 + \omega^i x_4 = x_2 + \omega^k x_3 = 0 \qquad (i, k = 0, 1, 2).$$

Since the lines meet by threes through the intersection of a_{00} and c_{00} passes a line L_{00}. The quadric passing through this point of intersection $(1, 1, 1, 1,)$ and having L_1, L_2 and L_5, L_6 as opposite sets of generators is $x_1 x_4 = x_2 x_3$. The generator through $(1, 1, 1, 1)$ cuts L_5 in $(1, 0, 1, 0)$ and L_6 in $(0, 1, 0, 1)$. Similarly generators through the intersections of a_{00} and c_{11}, of a_{00} and c_{22} cut on L_5 the points $(\omega, 0, 1, 0)$; $(\omega^2, 0, 1, 0)$; on L_6 the points $(0, \omega, 0, 1)$; $(0, \omega^2, 0, 1)$ respectively. The joins of the sets determined thus on L_5 and L_6 can be written as

$$(14) \qquad b_{ik}: \quad x_1 - \omega^i x_3 = x_2 - \omega^k x_4 = 0 \qquad (i, k = 0, 1, 2).$$

If we join the cubicovariant sets of these points on L_5 and L_6 we obtain the 9 lines

$$(15) \qquad b'_{ik}: \quad x_1 + \omega^i x_3 = x_2 + \omega^k x_4 = 0 \qquad (i, k = 0, 1, 2).$$

Now the 27 lines a'_{ik}, b'_{ik}, c'_{ik} are lines of the cubic surface

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0.$$

Similarly the sets of 27 lines

$$a'_{ik}, b_{ik}, c_{ik}; \quad a_{ik}, b'_{ik}, c_{ik}; \quad a_{ik}, b_{ik}, c'_{ik}$$

are lines of the surfaces

$$x_1^3 + x_2^3 - x_3^3 - x_4^3 = 0; \qquad x_1^3 - x_2^3 + x_3^3 - x_4^3 = 0;$$
$$\text{and} \quad x_1^3 - x_2^3 - x_3^3 + x_4^3 = 0 \text{ respectively.}$$

Further, if we join the points p_i on L_1 to the cubicovariant points q'_i on L_2, and similarly for the other 5 sets, we shall obtain a second set of 54 lines, 108 points and 108 planes possessing the same properties as the first set. We shall write the equations of the 6 sets of 9 lines each as

$$\begin{aligned}
A_{ik}&: \quad x_1 - \omega^i x_2 = x_3 + \omega^k x_4 = 0\\
A'_{ik}&: \quad x_1 + \omega\, x_2 = x_3 - \omega^k x_4 = 0\\
B_{ik}&: \quad x_1 - \omega^i x_3 = x_2 + \omega^k x_4 = 0\\
B'_{ik}&: \quad x_1 + \omega^i x_3 = x_2 - \omega^k x_4 = 0\\
C_{ik}&: \quad x_1 - \omega^i x_4 = x_2 + \omega^k x_3 = 0\\
C'_{ik}&: \quad x_1 + \omega^i x_4 = x_2 - \omega^k x_3 = 0 \qquad (i, k = 0, 1, 2).
\end{aligned}$$

The sets of 27 lines A_{ik}, B_{ik}, C_{ik}; A_{ik}, B'_{ik}, C'_{ik}; A'_{ik}, B_{ik}, C'_{ik}; and A'_{ik}, B'_{ik}, C_{ik} are lines of the surfaces

$$-x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0, \quad x_1^3 - x_2^3 + x_3^3 + x_4^3 = 0,$$
$$x_1^3 + x_2^3 - x_3^3 + x_4^3 = 0, \quad \text{and } x_1^3 + x_2^3 + x_3^3 - x_4^3 = 0$$

respectively.

5. *Quadric Surfaces on the Sets of* 54 *Lines.* There are nine quadric surfaces associated with the configuration such that each quadric contains 12 of the 54 lines a_{ik}, b_{ik}, c_{ik}, a'_{ik}, b'_{ik}, c'_{ik} $(i, k = 0, 1, 2)$. Starting with any line as a_{00}, we choose two other a_{ik} which do not meet it; we can choose either a_{11}, a_{22} or a_{12}, a_{21}. The other lines are then determined for with a_{00}, a_{11}, a_{22} are associated a'_{00}, a'_{11}, a'_{22}, b_{00}, b_{11}, b_{22}, b'_{00}, b'_{11}, b'_{22} which lie on the quadric surface $x_1x_4 = x_2x_3$; with a_{00}, a_{12}, a_{21}, are associated a'_{00}, a'_{12}, a'_{21}, c_{00}, c_{11}, c_{22}, c'_{00}, c'_{11}, c'_{22} which lie on the quadric $x_1x_3 = x_2x_4$. The number of quadrics will be $54 \cdot 2/12 = 9$. They are

$$(16) \qquad x_1x_4 = \omega^i x_2x_3, \quad x_1x_3 = \omega^i x_2x_4, \quad x_1x_2 = \omega^i x_3x_4 \qquad (i = 0, 1, 2).$$

Similarly the 54 lines A_{ik}, B_{ik}, C_{ik}, A'_{ik}, B'_{ik}, C'_{ik} $(i, k = 0, 1, 2)$ have associated with them nine quadric surfaces such that each surface contains 12 of the 54 lines as generators. The equations of the nine surfaces are

(17) $x_1x_4 = -\omega^i x_2x_3, \quad x_1x_3 = -\omega^i x_2x_4, \quad x_1x_2 = -\omega^i x_3x_4 \quad (i = 0, 1, 2).$

These eighteen quadrics are members of three pencils; the parameters of the pencil giving the equations of the quadrics in each case are the cube roots of positive and negative unity.

6. *Desmic Tetrahedra.* The whole configuration of 108 lines, 216 points and 216 planes can best be pictured by noting that it forms 54 tetrahedra, the lines being the edges, the points being the vertices and the planes being the faces. The 54 tetrahedra fall naturally into 27 sets of two each, such that each set of two together with the reference tetrahedron forms a desmic set, i. e.: they are four-fold perspective. Thus the 24 points of the configuration on the four lines a_{00}, a'_{00}, A_{00}, A'_{00} fall into three sets of eight, such that each set with the reference tetraherdon is a desmic set; moreover the 24 planes of the configuration on the four lines are the faces of the tetrahedra. Thus on the 36 lines a_{ik}, a'_{ik}, A_{ik}, A'_{ik} $(i, k = 0, 1, 2)$ are all 216 points and 216 planes forming the 27 sets of desmic tetrahedra, and the remaining 72 lines from the other edges of the tetrahedra. Similarly on the 36 lines b_{ik}, b'_{ik}, B_{ik}, B'_{ik} and on the 36 lines c_{ik}, c'_{ik}, C_{ik}, C'_{ik} $(i, k = 0, 1, 2)$ are all 216 points and 216 planes giving the same 27 sets of desmic tetrahedra.

The triple tangent planes of an Eckardt surface consist of 18 planes in which the three lines of the surface meet in a point, and 27 planes in which the lines do not meet in a point. The 216 planes of the configuration are the triple tangent planes of the latter type of the eight Eckardt surfaces.

Any set of eight points which together with the reference tetrahedron forms a desmic set bears a simple relationship to the eight Eckardt surfaces. As the eight points separate into two tetrahedra, the eight surfaces fall into two sets of four; each set associated with one tetrahedron. Through each vertex of the tetrahedron pass three of the four surfaces in the set, and each vertex with its opposite face are pole and polar plane as to the one surface of the set not passing through that vertex.

7. *The Group of the Configuration.* An Eckardt surface is unaltered by a collineation group of order 648; each collineation leaves the reference tetrahedron invariant. The surface is determined when the reference tetrahedron is given and one triple tangent plane in which the lines of the surface do not meet in a point.* The G_{648} consists of the product of a G_{27}, which

* Bobek, *Monatshefte für Mathematik*, Vol. 10 (1889), p. 156.

sends the given triple tangent plane into each of the 27 planes, and a G_{24}, which permutes the reference tetrahedron in all possible ways. If we add to this the G_8

$$(18) \qquad x_1' = x_1; \quad x_2' = \pm x_2; \quad x_3' = \pm x_3; \quad x_4' = \pm x_4,$$

which sends any one Eckardt surface into each of the eight surfaces we have the group of the configuration, an imprimitive group of order 5184.

The G_{5184} consists then of the product of a G_{24}, which permutes the reference tetrahedron in all possible ways, and a G_{216} which sends any one point (or plane) into the set of 216 points (or planes). The 18 quadric surfacs are conjugate under the group, each being invariant under a G_{288}. The 27 sets of desmic tetrahedra are conjugate under the G_{5184}; each is unaltered by a G_{192}. The permutative group on the 27 sets of desmic tetrahedra is

$$x_1' = x_1; \quad x_2' = \omega^i x_2; \quad x_3' = \omega^j x_3; \quad x_4' = \omega^k x_4; \qquad (i, j, k = 0, 1, 2).$$

8. *The Point-Plane Duality.* On each edge of the reference tetrahedron are six points. The six planes, one through each point and the opposite edge, contain all 216 points of the configuration, 36 in each plane. There are thus six systems, of six planes each, containing the 216 points. It can be seen easily that the 36 points in any plane lie by six on 18 lines, the lines passing by six through the two vertices of the tetrahedron in the plane, and through the given point on the opposite edge. The 36 points fall into four sets of nine each, such that each set together with the above mentioned three tetrahedron points are the vertices of the flex triangles of a pencil of cubic curves. The configuration in such a plane is invariant under an imprimitive ternary G_{216}.*

Similarly all 216 planes of the configuration are on the six points on any edge of the tetrahedron, 36 on each point. There are thus six sets, of six points each, which have on them all 216 planes. It can be seen also that the 36 planes on any point lie by six on 18 lines, the lines lying by six in three planes,—the two faces of the tetrahedron on the point and the plane on the point through the opposite edge.

9. *The Imprimitive Ternary G_{216}.* On any two sides of the reference triangle in the plane choose a set of points such that the vertices of the triangle on those sides are the Hessian points of the sets. Construct on both

* This G_{216} is discussed in the following section.

sides also the cubicovariant points of each given set. Join now the six points on one side with the six points on the second side in all possible ways. These 36 lines will cut the third side of the triangle in six points having the same relation to it as the originally chosen sets have to the sides on which they are located. We have thus a set of 18 points and 36 lines such that three points are on each line and six lines on each point. The 36 lines will fall naturally into four sets of nine each, such that each set together with the reference triangle are the flex triangles of a pencil of cubic curves.

If we designate by a_i the points $0, \omega^i, 1$; by b_i the points $\omega^i, 0, 1$; by c_i the points $\omega^i, 1, 0$; $(i = 0, 1, 2)$ and the cubicovariant sets of the above respectively as a_i', b_i' and c_i' then

a_i', b_i', c_i' are the flexes of $x_1^3 + x_2^3 + x_3^3 + 6mx_1x_2x_3 = 0$

a_i', b_i, c_i are the flexes of $-x_1^3 + x_2^3 + x_3^3 - 6mx_1x_2x_3 = 0$

a_i, b_i', c_i are the flexes of $x_1^3 - x_2^3 + x_3^3 - 6mx_1x_2x_3 = 0$

a_i, b_i, c_i' are the flexes of $x_1^3 + x_2^3 - x_3^3 - 6mx_1x_2x_3 = 0.$

These 36 lines intersect in 36 points, not including the original points on the sides of the reference triangle. They are the vertices of the flex triangles of the four pencils of cubics. As such they must also lie on the harmonic polars of the cubics. We obtain thus a set of 36 points and 18 lines, such that three lines are on each point and six points on each line, the lines passing by six through the vertices of the reference triangle. This is the dual configuration of the 18 points and 36 lines.

The configuration consists of four Hesse configurations with a common triangle—the reference triangle which is invariant under the group. The G_4 $x_1' = x_1$; $x_2' = \pm x_2$; $x_3' = \pm x_3$ will send any one of the four Hesse configurations into each other so they are conjugate under the group. A Hesse configuration has associated with it a Hesse G_{216},* which contains a G_{54} leaving one flex triangle invariant. Hence the group of this configuration is the product of this G_{54} and the above G_4 which gives a G_{216}, obviously imprimitive.

A Hesse configuration is determined when one flex triangle, say the reference triangle, is given, and one point say $(1, 1, 1)$ of another flex triangle. The above configuration consequently is determined when the reference tri-

* A detailed analysis of this group has been recently given by Winger, *University of Washington Publications*, June 1926.

angle is given as the common flex triangle and the four points $(1, \pm 1, \pm 1)$. Now the imprimitive ternary G_{216} consists of the product of a G_9: $x_1' = x_1$; $x_2' = \omega^i x_2$; $x_3' = \omega^j x_3$ $(i, j = 0, 1, 2)$ which leaves each Hesse configuration invariant; and a G_{24} which permutes the four points or the four configurations in all possible ways.

If we take the general case of n Hesse configurations with a common triangle and all conjugate under a collineation group of order n generated by $x_1' = x_1$; $x_2' = \epsilon^i x_2$; $x_3' = \epsilon^j x_3$ where ϵ is an nth root of unity, the permutation group of these n Hesse configurations will have its maximum possible order when $n = 4$. Hence the above configuration is the one of most interest.

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Rational Involutorial Transformations in S_4 which Leave Invariant ∞^4 Quadric Varieties.

By Howard Conway Shaub.

1. *Introduction.* An involutorial transformation I, of a four dimensional space (x), is a birational transformation of period two. By I the points of (x) are associated in pairs P_1 and P_2, such that each is the image of the other in the transformation. If a linear space (x') can be found such that to a point P' of (x') corresponds the two associated points P_1 and P_2 of (x), then I is said to be rational. To a hyperplane in (x') corresponds a variety of the same dimensions in (x) which will be invariant under the involution. We shall consider the simplest case, namely when these varieties are quadrics. The methods employed are the natural extensions to S_4 of those developed in the classical memoirs of De Paolis for S_2 * and S_3.†

2. *Equivalence and Postulation of a Curve.* A point in (x') is determined as the intersection of four hyperplanes. The four corresponding quadric varieties must determine the two corresponding points in the involution, hence any four quadric varieties can have only two variable intersections. Four general quadric varieties intersect in 16 points. It follows that the system of quadric varieties corresponding to the hyperplanes of (x') must be of dimension four, and must have in common basis elements which absorb 14 of the 16 intersections. Since the dimension of the system of all the quadric varieties in S_4 is 14, the basis elements can not impose more than 10 conditions on the system.

The maximum number of simple basis points is 10, in which case there are 6 variable intersections. Therefore the basis system can not consist of points alone. A double point or a double line is also impossible. There must be, then, at least a basis curve C_m. If it is of order m and genus p its *postulation* for quadric varieties is

$$P_m = 2m - p + 1,$$

as in the case of ordinary space.

* R. De Paolis, "Le trasformazione piane doppie," *Atti delle Reale Accademia de Lincei*, ser. 3, Vol. 1 (1877), pp. 511-544.

† R. De Paolis, "Le trasformazione doppie dello spazio," *Lincei Memorie*, ser. 4, Vol 1 (1885), pp. 576-609.

The equivalence of C_m may be derived from the following theorem due to Caporali: *

THEOREM. *If three varieties of orders n_1, n_2, n_3, respectively, pass through the curve C_m, and if the residual curve of intersection of the 3 varieties meets C_m in t points, then $m(n_1 + n_2 + n_3 - 3) = r + t$, where $r = 2m + 2p - 2$ is the rank of C_m.*

Hence in our case $t = m - 2p + 2$. The order of the residual curve is $8 - m$. The residual curve meets a fourth quadric variety of the system in $16 - 2m - t$ points not on C_m. Hence the equivalence of C_m is

$$E_m = 2m + t = 3m - 2p + 2.$$

In finding possible sets of basis elements it will be convenient to denote by S the number of simple basis points; that is

$$S = 14 - E_m = 12 - 3m + 2p.$$

And we shall denote the number of linearly independent quadric varieties through the fixed basis elements by N. Then

$$N = 15 - P_m - S = 14 - 2m + p - S = 2 + m - p.$$

Any set of values of m, p, and S which satisfies the inequality $N \geqq 5$, and which leads to no geometrical contradiction, furnishes a set of basis elements for I. By varying m and p we obtain the following sets:

m	p	S	N
2	−1	4	5
3	0	3	5
3	−1	1	6
4	1	2	5
4	0	0	6
5	2	1	5
6	3	0	5

It will now be shown from geometrical considerations that the last set does not give a $(1, 2)$ correspondence. A C_6, $p = 3$, lies in an S_3 and not on a quadric surface. The 5 linearly independent quadric varieties through C_6 consist therefore of the S_3 and 5 linearly independent hyperplanes. Thus these quadric varieties define a linear correspondence instead of a $(1, 2)$ correspondence.

* E. Caporali, *Memorie di Geometria*, Naples (1888), p. 320.

3. *Equivalence and Postulation of a Plane.* Let

$$\pi \equiv x_1 = x_2 = 0,$$

be a fundamental plane. The most general form of a quadric through π is

$$V_2 \equiv b + cx_3 + dx_4 + ex_5 = 0,$$

where c, d, and e are linear in x_1 and x_2, and b is quadratic in the same variables. The left hand member contains 9 homogeneous constants, therefore the postulation of a plane for a quadric variety is 6. To find the residual intersection of 3 quadric varieties through π, take a variable hyperplane $x_1 = \lambda x_2$ through π. This hyperplane meets V_2 in a plane given by

$$x_1 = \lambda x_2,$$
$$H \equiv b(\lambda)x_2 + c(\lambda)x_3 + d(\lambda)x_4 + e(\lambda)x_5 = 0,$$

where c, d, and e are linear in λ, and b is qudaratic in λ. The residual curve of intersection of 3 quadric varieties through π is the locus of intersections of the 4 hyperplanes

$$x_1 = \lambda x_2,$$
$$H_i = b_i(\lambda)x_2 + c_i(\lambda)x_3 + d_i(\lambda)x_4 + e_i(\lambda)x_5, \qquad (i = 1, 2, 3).$$

From the matrix of the coefficients

$$\left\| \begin{matrix} -1 & \lambda & 0 & 0 & 0 \\ 0 & b_1(\lambda) & c_1(\lambda) & d_1(\lambda) & e_1(\lambda) \\ 0 & b_2(\lambda) & c_2(\lambda) & d_2(\lambda) & e_2(\lambda) \\ 0 & b_3(\lambda) & c_3(\lambda) & d_3(\lambda) & e_3(\lambda) \end{matrix} \right\|$$

we have the parametric equations of the residual curve in the form

$$x_1 = \lambda\theta_3(\lambda), \quad x_2 = \theta_3(\lambda), \quad x_3 = \phi_4(\lambda), \quad x_4 = \psi_4(\lambda), \quad x_5 = \chi_4(\lambda);$$

where, on the right, the suffix denotes the degree in λ. Thus we see that the residual intersection of 3 quadric varieties through π is a rational C_4, which meets π in the points corresponding to the 3 roots of $\theta_3(\lambda) = 0$. A fourth quadric variety meets the residual C_4 in 8 points. Since 3 of these points are on π, only 5 of them are variable, showing that the equivalence of a plane is 11. Hence for this case

$$S = 3, \quad N = 6.$$

Consider next the case of a fundamental plane and a line meeting the plane. Let the plane be

$$\pi \equiv x_1 = x_2 = 0;$$

and let the line be

$$l \equiv x_1 = x_3 = x_4 = 0.$$

The most general quadric variety through π and l has an equation of the form

$$V_2 \equiv ax_1 + cx_3 + dx_4 + ex_1x_5 = 0,$$

where a, c, and d are linear in x_1 and x_2, and e is a constant. There are 7 homogeneous constants, hence the postulation of π and l together is 8. The equation of the residual curve of intersection of 3 quadric varieties through π and l is obtained as in the preceding case. The coordinates of points on this curve, when expressed in terms of the elements of the matrix

$$\begin{Vmatrix} -1 & \lambda & 0 & 0 & 0 \\ 0 & b_1(\lambda) & c_1(\lambda) & d_1(\lambda) & e_1 \\ 0 & b_2(\lambda) & c_2(\lambda) & d_2(\lambda) & e_2 \\ 0 & b_3(\lambda) & c_3(\lambda) & d_3(\lambda) & e_3 \end{Vmatrix}$$

are of the form

$$x_1 = \lambda\theta_2(\lambda), \quad x_2 = \theta_2(\lambda), \quad x_3 = \lambda\phi_2(\lambda), \quad x_4 = \lambda\psi_2(\lambda), \quad x_5 = \chi_2(\lambda).$$

The residual curve is therefore a cubic which meets π in two points corresponding to the roots of $\theta_2(\lambda) = 0$, and also meets l in one point given by $\lambda = 0$. A fourth quadric variety through π and l meets C_3 in 6 points, two of which are on π and one on l, leaving 3 variable intersections. Thus the combined equivalence of π and l is 13, and we have

$$S = 1, \quad N = 6.$$

Consider next the case where l does not meet π. Let

$$\pi \equiv x_1 = x_2 = 0;$$

and

$$l \equiv x_3 = x_4 = x_5 = 0.$$

Then the most general quadric variety through π and l has an equation of the form

$$cx_3 + dx_4 + ex_5 = 0,$$

where c, d, and e are linear and binary in x_1 and x_2. There are 6 homogeneous constants, hence the postulation for π and l together is 9. As in

the previous case, we cut 3 quadric varieties through π and l by a hyperplane $x_1 = \lambda x_2$. The 3 resulting hyperplanes have the equations

$$c_i(\lambda)x_3 + d_i(\lambda)x_4 + e_i(\lambda)x_5 = 0, \qquad (i = 1, 2, 3),$$

which have solutions for the 3 values of λ given by

$$\Delta_3(\lambda) \equiv \begin{vmatrix} c_1(\lambda) & d_1(\lambda) & e_1(\lambda) \\ c_2(\lambda) & d_2(\lambda) & e_2(\lambda) \\ c_3(\lambda) & d_3(\lambda) & e_3(\lambda) \end{vmatrix} = 0.$$

For each of these values λ_i, $(i = 1, 2, 3)$, the 3 hyperplanes have a plane through l in common which meets the hyperplane $x_1 = \lambda x_2$ in a line. Therefore the residual C_3 consists of 3 lines meeting π and l. A fourth quadric variety through π and l meets these three lines only on π and l. There are, therefore, no variable intersections; that is the equivalence of π and l is 16. Hence a plane and a line not meeting the plane do not furnish a set of basis elements for a $(1, 2)$ correspondence.

4. *Equivalence and Postulation of a Quadric Surface.* Let

$$\Gamma_2 \equiv f_2(x_1, x_2, x_3, x_4) = x_5 = 0,$$

be a fundamental quadric surface. The most general form of a quadric variety through Γ_2 is

$$Af_2(x_1, x_2, x_3, x_4) + x_5(ax_1 + bx_2 + cx_3 + dx_4 + ex_5) = 0.$$

The equation contains 6 homogeneous constants, hence the postulation of Γ_2 is 9. Three quadratic varieties

$$Af_2 + u_i x_5 = 0. \qquad (i = 1, 2, 3),$$

through Γ_2 have as a residual intersection a conic C_2 determined by the plane $\pi \equiv u_1 = u_2 = u_3 = 0$ and the quadric cone $Af_2 + u_1 x_5 = 0$. The plane π meets Γ_2 in 2 points A, B, which must lie on C_2 since C_2 contains all the points common to π and $Af_2 + u_1 x_5 = 0$. Hence the residual intersection of 3 quadric varieties through Γ_2 is a conic meeting Γ_2 in two points. A fourth quadric variety through Γ_2 meets C_2 in 4 points, 2 of which must be A and B. The remaining two intersections P_1, P_2, vary with the set of four quadrics and are the images of the intersection P' of the 4 corresponding hyperplanes in (x'). Hence the equivalence of Γ_2 is 14, giving the values

$$S = 0, \quad N = 6.$$

The line P_1P_2 passes through a fixed point O common to all the hyperplanes which together with $H \equiv x_5 = 0$ constitute composite quadrics of the system; the corresponding hyperplanes in (x') have a point O' in common.

As a special case of a fundamental quadric surface we have two planes meeting in a line. But if the two planes meet only in a point the postulation is 11; consequently these two planes cannot make up a fundamental system for a $(1, 2)$ correspondence.

A cubic surface σ_3 will not serve for a basis element. For if σ_3 lies in an S_3 it does not lie on a quadric variety because the section of a quadric variety by an S_3 is a quadric surface. Again if σ_3 is the residual intersection of two quadric varieties through a plane then it does not lie in an S_3. Hence the number of linearly independent quadric varieties through σ_3 is the same as the number of linearly independent quadrics through a space cubic, namely 3. Since N must be at least 5, the quadric varieties through σ_3 do not furnish a $(1, 2)$ correspondence.

5. *Sets of Basis Elements.* We have found, then, the following sets of basis elements for a $(1, 2)$ correspondence:

a). C_2, $p = -1$, and 4 points,

b). C_3, $p = 0$, and 3 points,

c). C_3, $p = -1$, and 1 point,

d). C_4, $p = 1$, and 2 points,

c_2). C_4, $p = 0$,

c_3). C_5, $p = 2$, and 1 point,

c_4). A plane, and 3 points not on the plane,

c_5). A plane, a line meeting the plane, and a point not on the line or the plane,

e). A quadric surface.

Case c_3) is equivalent to case e). Since $p = 2$, C_5 lies in a hyperplane and hence must lie on all the quadrics which are the sections of the quadric varieties of the system by this hyperplane. But since the postulation of a C_5 for a quadric is 9, there is just one quadric which contains C_5. Let this quadric be

$$Q \equiv x_5 = \phi_2(x_1, x_2, x_3, x_4) = 0.$$

The general equation of a quadric variety has the form

$$f_2(x_1, x_2, x_3, x_4) + x_5 \sum_1^5 a_i x_i = 0.$$

But since Q is the only quadric containing C_5, the above equation must reduce to the form

$$\phi_2(x_1, x_2, x_3, x_4) + x_5 \sum_1^5 a_i x_i = 0.$$

Thus if the quadric variety contains C_5 it must also contain the quadric surface on which C_5 lies, that is we have a fundamental quadric surface.

Cases c_i), $(i = 2, 4, 5)$ can be reduced by a birational transformation to case e). For in the set of ∞^5 quadric varieties there are ∞^4 through an additional point O, which determine a birational transformation T. Of the ∞^4 quadric varieties used in defining the $(1, 2)$ correspondence, ∞^3 pass through O and are transformed by T into quadric varieties composed of a fixed hyperplane,—the image of O,—and a variable hyperplane. Any quadric variety of the system, not through O, is transformed into another quadric variety. It follows that the ∞^4 quadric varieties used in defining the $(1, 2)$ correspondence are transformed into quadric varieties through a fixed quadric surface, namely the intersection of the hyperplane which is the image of O and the image of any quadric variety of the system not through O. Hence the cases c_i), $(i = 2, 4, 5)$ are transformable into case e).

There are therefore, 4 independent types of systems of quadric varieties, Σ, which determine $(1, 2)$ correspondences, namely those with the following basis elements:

I. Quartic curve β_4, $p = 1$, and 2 simple points P_1, P_2.

II. Cubic curve β_3, $p = 0$, and 3 simple points P_1, P_2, P_3.*

III. Two skew lines α_1, β_1, and 4 simple points P_1, P_2, P_3, P_4.

IV. Quadric surface Γ_2.

We proceed to determine the involution I associated with each of these types.

6. *Type I. Basis system a quartic curve β_4, $p = 1$, and 2 simple points P_1, P_2.* The image of a hyperplane V'_1 in (x') is a quadric variety V_2 in (x), passing through β_4, P_1, P_2. A plane π' in (x'), being the intersection of 2 hyperplanes, has for its image in (x) a quartic surface which is the intersection of the 2 quadric varieties corresponding to the 2 hyperplanes. A line l' in (x'),—the intersection of 3 hyperplanes,—has for its image a quartic

* G. Aprile, "Di una trasformazione doppia nello spazio a quattro dimensioni," *Giornale di Matematiche* (3), Vol. 56 (1918), pp. 90-140 and Vol. 57 (1919), pp. 129-149.

curve C_4, $p = 1$, the residual intersection of the 3 quadric varieties corresponding to the 3 hyperplanes. The curve C_4 passes through P_1, P_2, and meets β_4 in 4 points.

The order of the image in (x') of a hyperplane V_1, in (x), is the number of points in which this image meets a line. To each of these points corresponds, in (x), an intersection of a C_4,—the image of the line,—with V_1. The image of V_1 is therefore a quartic variety in (x'). The order of the image in (x') of a plane π, in (x), is the number of points in which this image meets an arbitrary plane π'. To each of these points corresponds, in (x), an intersection of π with a quartic surface,—the image of π'. The image of π is therefore a quartic surface in (x'). The order of the image in (x') of a line l, in (x), is the number of points in which this image meets an arbitrary hyperplane V_1'. To each of these points there corresponds, in (x), an intersection of a quadric variety,—the image of V_1',—with l. The image of l is therefore a conic in (x').

The hyperplanes in (x') whose images are composed of the quadric varieties of Σ which contain the line λ through P_1, P_2, form a web, and therefore have in common a point O' which is the image of λ.

The hyperplanes in (x') whose images are composed of the hyperplane H in which β_4 lies and a hyperplane through λ, form a net (a subset of the system Σ), and therefore have in common a line λ' through O', which is the image of H. Among the quadric varieties of Σ there is a pencil which does not contain the hyperplane H. A general hyperplane in (x') meets λ' in a point to which corresponds in (x) the intersection of H with one of the quadric varieties of the pencil. The image of a point on λ' is therefore a quadric surface lying in H.

Since the net of hyperplanes through λ' has for its image in (x) the net of hyperplanes through λ and the common hyperplane H, a plane through λ' has for its image a plane through λ. A general hyperplane V_1' meets the plane through λ' in a line whose image is the conic through P_1 and P_2 in which the image of V_1' meets the plane through λ. A general hyperplane V_1, in (x), meets this conic in 2 points; hence a line meeting λ' cuts V_4', the quartic image of V_1, in 2 points not on λ'. A line meeting λ' must therefore meet V_4' in 2 points on λ'; hence λ' is double on V_4'. The complete image of V_4' is a V_8: β_4^4, P_1^2, P_2^2, having as components the original V_1 and H taken twice. The conjugate in I of an arbitrary V_1 is therefore a V_5: β_4^2, P_1^4, P_2^4.

We have seen that the image of a point P' on λ' is a quadric surface of the pencil in H through β_4. Through a point R_1 of H passes a surface Q

of the pencil (the image of a point R' on λ') and a definite plane π through λ. To the plane π corresponds a plane π' through λ'. The image of R_1 is a direction through R' in π'. The image of R' is R_1 and R_2, the residual intersection of Q and π. Hence the image in I of R_1 is R_2, and H is invariant under I.

Corresponding to each of the 4 points in which β_4 meets a C_4,—the image of a line l',— is an intersection of l' with the image of β_4; the image in (x') of β_4 is therefore a quartic variety V_4. Since a general plane thru λ does not meet β_4, a line meeting λ' does not meet V_4 except on λ'; hence λ' is fourfold on V_4'. The image of a plane π through λ and a point P of β_4 is a plane π' through λ'. The images of the lines of intersection with π' of two hyperplanes through an arbitrary point P' of π' are two conics in π through P_1, P_2, P, and Q, a fourth point of π. Thus the images in (x) of P' are P and Q, and hence the image of π' is π and P. As P moves along β_4, π generates a cone K_4 having the line λ for vertex, and π' traces out a similar cone K_4' with the line vertex λ'. That is, the image of β_4 is K_4', but the image of K'_4 is K_4 and β_4. Hence in I the image of β_4 is K_4, which obviously contains β_4.

In general, to a point P' of (x') there correspond two points P_1 and P_2 of (x); but for certain positions of P' the points P_1 and P_2 coincide. The locus of such points P' is the variety of branch points, L'; the locus of the points P_1, P_2, is the variety of coincidences K; that is, K taken twice, is the complete image of L'. The jacobian of the system Σ is the locus of the vertices of the quadric varieties of Σ which are cones. The hyperplanes through a point of L' have for images quadric varieties in (x) with a common tangent line at the corresponding point of K. Hence among them is a cone with this point as vertex, that is K is part of the jacobian of Σ.

If we apply Zeuthen's * formula for the number of coincidences in a multiple correspondence to the $(1; 2)$ correspondence between the points of the line C_1' in (x') and its image C_4, $p = 1$ in (x), we find that there are 4 points on C_1' such that their 2 images in (x) coincide. Thus L' is of order 4. Applying the same formula to a line C' which meets λ' and has therefore a conic for image in (x), we find that there are 2 points on such a line, apart from points on λ', such that their images in (x) coincide. Thus λ' must be double on L_4'. Since the image of λ' is H, and the image of L_4' is K^2, then K must be of order 3, must contain the fundamental curve β_4, and have P_1

* H. G. Zeuthen, *Mathematische Annalen*, Vol. 3 (1871), pp. 150-156.

and P_2 for double points. The jacobian is of order 5, has β_4 for a triple curve, and has K_3 for one component. The residual is therefore a quadric variety having β_4 for a double curve and hence must consist of H taken twice, since β_4 is simple on any noncomposite quadric variety containing it.

To the ∞^3 directions thru the fundamental point P_1 in (x) correspond the ∞^3 points of some variety V', in (x'). The order of V' is the number of points in which it is met by a line l'. But since the C_4 in (x) which is the image of l', passes through P_1 in a single direction, then l' meets V' in just one point; that is the image of P_1 is a hyperplane H_1'. To a point on H_1' there corresponds in (x) a direction through P_1 and a point on the quadric variety of Σ which is the image of H_1'. If the point on H_1' is also on L_4' then both the images of the point are directions at P_1 and lie on the tangent cone to K_3 at P_1. Therefore the image of P_1 in I is the quadric cone of Σ which has P_1 for a vertex. Similarly the image of the point P_2 in I is the quadric cone of Σ which has P_2 for a vertex.

Collecting our results, we have an involution I in which

$$
\begin{aligned}
&V_1 \sim V_5\colon\ \beta_4^2,\ P_1^4,\ P_2^4,\\
&\beta_4 \sim K_4\colon\ \beta_4,\ P_1^4,\ P_2^4,\\
&P_1 \sim V_2\colon\ \beta_4,\ P_1^2,\ P_2,\\
&P_2 \sim V_2\colon\ \beta_4,\ P_1,\ P_2^2,\\
&H \sim H.\\
&K_3\colon\ \beta_4,\ P_1^2,\ P_2^2.
\end{aligned}
$$

The $(1,2)$ correspondence between the spaces (x') and (x) can be expressed algebraically in the form

$$
\begin{aligned}
&x_1' = x_5x_1, \quad x_2' = x_5x_2, \quad x_3' = x_5x_3,\\
&x_4' = u_1x_4 + u_2, \quad x_5' = x_4^2 - x_5^2 + v_1x_4 + v_2,
\end{aligned}
$$

where the suffixes of u and v indicate the degree in x_1, x_2, x_3; and from these equations all the results obtained can be deduced algebraically. The fundamental elements have the following equations:

$$
\begin{aligned}
&H\colon\ x_5 = 0; \quad \lambda\colon\ x_1 = x_2 = x_3 = 0; \quad \lambda'\colon\ x_1' = x_2' = x_3' = 0;\\
&\beta_4\colon\ x_5 = 0; \quad u_1x_4 + u_2 = 0; \quad x_4^2 + v_1x_4 + v_2 = 0;\\
&K_4\colon\ u_2^2 - u_1u_2v + u_1^2v_2 = 0.
\end{aligned}
$$

7. *Type II. Basis system a cubic curve β_3, $p = 0$, and 3 simple points P_1, P_2, P_3.* As in the preceding case, the image of a hyperplane V_1', in (x'),

is a quadric variety V_2, in (x), passing through β_3, P_1, P_2, and P_3. The image of a plane π' is a quartic surface in (x) also passing through β_3, P_1, P_2, and P_3. A line l_1' has for its image a quintic curve, $p = 1$, passing through P_1, P_2, and P_3, and meeting β_3 in 5 points.

To each of the points in which the image of a V_1 meets an arbitrary line of (x') there corresponds an intersection of a C_5 with V_1. Hence the image of V_1 is a quintic variety V'_5. To each of the points in which the image of a plane π of (x) meets an arbitrary plane of (x') there corresponds an intersection of a quartic surface with π. Hence the image of π is a quartic surface in (x'). To each of the points in which the image of a line l of (x) meets an arbitrary V_1' there corresponds an intersection of a V_2 of (x) with l. Hence the image of l is a conic in (x').

The hyperplanes of (x') whose images are composed of the space H in which β_3 lies and a hyperplane through the plane π determined by P_1, P_2, and P_3, form a pencil and hence have a plane π' in common which is the image of H and π. The system of hyperplanes Σ' meets π' in a net of lines. A V_1' has for image a V_2 which meets H in a quadric Q_2 through β_3, the image of the line in which V_1' meets π'. The V_2 meets π in a conic through P_1, P_2, and P_3, which is conjugate to Q_2 in the involution I. Two lines of π' meet in a point P' whose images in (x) are the residual intersection P of the 2 corresponding conics of π, and the bisecant p of β_3 which is the residual intersection of the 2 corresponding quadrics of H. As the point P' traces out π', the bisecant of β_3 describes H, and P describes π. An arbitrary V_1 meets p in one point, hence P' is a simple point on V_5', the image of V_1, and the plane π' is simple on V_5'. The complete image of V_5' is a V_{10}: β_3^5, P_1^5, P_2^5, P_3^5, having the original V_1 and H as components. The conjugate of an arbitrary V_1 in I is therefore a V_8: β_3^4, P_1^5, P_2^5, P_3^5, π.

Since the image of an arbitrary line l' is a C_5 passing simply through P_1, the image of P_1 must be a variety meeting l' in only one point, that is a hyperplane. Hence the images of the fundamental points P_1, P_2, P_3, are hyperplanes H_1', H_2', H_3' respectively. By an argument similar to that used for P_1 and P_2 in Case I the images of H_1', H_2', and H_3' can be shown to be the quadric cones of Σ whose vertices are P_1, P_2, and P_3 respectively; that is, these cones are the images in I of the fundamental points P_1, P_2, and P_3 respectively.

Corresponding to each of the 5 points in which β_3 meets a C_5, the image of an arbitrary line l' of (x'), is an intersection of l' with the image of β_3. Hence the image in (x') of β_3 is a quintic variety B_5'. But the image of a line l' meeting π' in a point P' is composite, consisting of a bisecant of β_3,—

the image of P',—and a C_4 meeting β_3 in 3 points. So l' meets B_5' in 3 points not on π', and hence π' is double on B_5'. The image in (x) of an arbitrary V_5' is a V_{10}: β_3^5, P_1^5, P_2^5, P_3^5. But since π' is double on B_5', the V_{10} has H as a double component, so that the conjugate of β_3 in I is a variety of order eight, B_8: P_1^5, P_2^5, P_3^5. The image of an arbitrary line in (x) is a conic in (x') meeting π'. The image of a line l meeting β_3 but not lying in H is composite, consisting of 2 lines meeting in a point P. One of these lines lies in B_5'; the other meets B_5' in 4 points apart from P which are the images of the points, not on β_3, in which l meets B_8. Hence β_3 is fourfold on B_8.

On the jacobian of Σ, which is of order 5, β_3 is triple and the points P_1, P_2, and P_3 are double. The hyperplane H, being fundamental, is part of the jacobian. The residual is the variety of coincidences K_4: β_1^2, P_1^2, P_2^2, P_3^2. Applying Zeuthen's formula to the $(1,2)$ correspondence between points on the line l' and its image C_5, $p=1$, we find that there are 4 points on l' such that their images in (x) coincide. Hence the variety of branch points L_4' is of order 4. A line l' meeting π' in P' has for image a C_4 meeting H in 3 points on β_3 and passing through P_1, P_2, and P_3 on π, and hence meeting K_4 in 4 points not on the images of π'. Therefore L_4' meets l' in 4 points apart from π'; that is L_4' does not contain π'.

Collecting our results, we have an involution I in which

$$V_1 \sim V_8\ \beta_3^4,\ P_1^5,\ P_2^5,\ P_3^5,\ \pi.$$
$$\beta_3 \sim V_8\colon\ \beta_3^4,\ P_1^5,\ P_2^5,\ P_3^5.$$
$$P_1 \sim V_2\colon\ \beta_3,\ P_1^2,\ P_2,\ P_3.$$
$$P_2 \sim V_2\colon\ \beta_3,\ P_1,\ P_2^2,\ P_3.$$
$$P_3 \sim V_2\colon\ \beta_3,\ P_1,\ P_2,\ P_3^2.$$
$$\pi \sim H.$$
$$K_4\colon\ \beta_3^2,\ P_1^2,\ P_2^2,\ P_3^2.$$

8. *Type III. Basis system 2 straight lines α_1, β_1, and 4 simple points P_1, P_2, P_3, P_4.* As in types I and II the image of a hyperplane V_1' in (x') is a quadric variety V_2 in (x) passing through α_1, β_1, P_1, P_2, P_3, P_4. The image of a plane π' is a quartic surface in (x) also passing through α_1, β_1, P_1, P_2, P_3, P_4. A line l' has for its image in (x) a sextic curve C_6, $p=1$, passing through P_1, P_2, P_3, P_4, and meeting α_1 and β_1 each in 3 points.

To each of the points in which the image of a V_1 meets an arbitrary line

of (x') there corresponds an intersection of a C_6 with V_1. Hence the image of a V_1 is a sextic variety V_6'. To each of the points in which the image of a plane π meets an arbitrary plane of (x') there corresponds in (x) the intersection of a quartic surface with π. Hence the image of π is a quartic surface in (x'). To each of the points in which the image of a line l of (x) meets an arbitrary V_1' there corresponds, in (x), the intersections of a V_2 with l. Therefore the image of l is a conic in (x').

The hyperplane H determined by the lines α_1 and β_1 contains ∞^2 bisecants of α_1, β_1. Through a bisecant a passes a web of V_2's of Σ since every V_2 of Σ contains 2 points of a. To this web of V_2's corresponds the web of V_1''s of Σ' through a point (a'). Hence the image of a bisecant of α_1, β_1 is a point, and the image of H is a surface, Q'. The order of Q' is the number of points in which it is met by an arbitrary plane π'. The image of π' is a quartic surface which does not lie in H, since if it did π' would lie in Q'. H meets this quartic surface in a C_4 which is the intersection of two quadrics of H through α_1 and β_1. The intersection of two such quadrics is a curve of the second order meeting α_1, β_1 each in two points. But when the curve of intersection of 2 surfaces breaks up into 2 curves of the same order the curves also have the same genus. Therefore the residual intersection of the 2 quadrics is two bisecants of α_1, β_1. These bisecants correspond to two intersections of π' and Q', hence Q' is a quadric surface.

Let H' be the hyperplane which has for its image the composite V_2 of Σ consisting of H, and $\overline{H}$ (the hyperplane determined by the four points P_1, P_2, P_3, P_4). Obviously H' contains Q_2', which is the intersection by H' of any V_2' through Q_2'. The image of this V_2' consists of H, the image of Q_2', and a residual V_3. The image of H' is H and $\overline{H}$, and therefore the image of Q_2' is H and the cubic surface σ_3 in which $\overline{H}$ meets V_3. Thus the image of H in I is the cubic surface σ_3.

Since an arbitrary V_1 of (x) meets each bisecant of α_1, β_1 in a point, the V_6' which is the image of V_1 contains Q_2' simply. The image of V_6' is then a V_{12}: α_1^6, β_1^6, P_1^6, P_2^6, P_3^6, P_4^6, σ_3, having the original V_1 and H as components. The conjugate of an arbitrary V_1 in I is therefore a V_{10}: α_1^5, β_1^5, P_1^6, P_2^6, P_3^6, P_4^6, σ_3.

The image in (x) of a line in (x') is a C_6, $p=1$, meeting α_1 and β_1 each in 3 points. Hence the image in (x') of α_1 is a cubic variety A_3', and the image of β_1 is B_3'. The image of a line l' meeting Q' consists of a bisecant of α_1, β_1 and a C_5 meeting α_1, β_1 each in 2 points. Therefore l' meets A_3' and B_3' each in two points apart from Q_2' and so Q_2' must be simple on A_3' and B_3'. The image in (x) of an arbitrary V_3' is a V_6: α_1^3, β_1^3, P_1^3, P_2^3,

P_3^3, P_4^3. But since A_3' contains Q' its image must be a V_5: $\beta_1^2, P_1^3, P_2^3, P_3^3, P_4^3$. The image of a line l meeting H in a point of α_1 but not lying in H is a composite conic consisting of a line l_1' of A_3' and a line l_2' meeting l_1'. Then l_2' meets A_3' in 2 points apart from l_1' and therefore A_5 meets l in 2 points not on α_1. Hence α_1 is triple on A_5. We have then, that the image of α_1 in I is A_5: $\alpha_1^3, \beta_1^2, P_1^3, P_2^3, P_3^3, P_4^3$ and the image of β_1 is B_5: $\alpha_1^2, \beta_1^3, P_1^3, P_2^3, P_3^3, P_4^3$.

Since the image of an arbitrary line l' is a C_5 passing simply through P_1, the image of P_1 must be a variety meeting l' in only one point, that is a hyperplane. Hence the images of the fundamental points P_1, P_2, P_3, P_4 are the hyperplanes H_1', H_2', H_3' H_4' respectively. By an argument similar to that employed in Cases I and II the images of H_1' H_2', H_3', H_4' can be shown to be quadric cones of Σ whose vertices are P_1, P_2, P_3, P_4 respectively; that is these cones are the images in I of the fundamental points P_1, P_2, P_3, P_4 respectively.

The jacobian of Σ is J_5: $\alpha_1^3, \beta_1^3, P_1^2, P_2^2, P_3^2, P_4^2$. It contains H as a component. For let (a) be any point of H. Among the V_2 of Σ which contain the secant of α_1, β_1 through (a) is a cone with (a) as vertex. Hence the residual variety of coincidences is K_4: $\alpha_1^2, \beta_1^2, P_1^2, P_2^2, P_3^2, P_4^2$. Applying Zeuthen's formula to the $(1, 2)$ correspondence between points on the line l' and its image C_6, $p = 1$, we find that there are 4 points on l' such that their images on C_6 coincide. Hence the variety of branch points L_4' is of order 4. A line l' meeting Q_2' in a point P' has for image a C_5 meeting α_1 and β_1 each in two points and passing through P_1, P_2, P_3, P_4, and hence meeting K_4 in 4 points not on the image of Q_2'. Therefore l' meets L_4' in 4 points apart from Q_2', and so Q_2' is not on L_4'.

Collecting our results, we have an involution I in which

$$V_1 \sim V_{10}: \alpha_1^5, \beta_1^5, P_1^6, P_2^6, P_3^6, P_4^6, \sigma_3.$$

$$\alpha_1 \sim A_5: \alpha_1^3, \beta_1^2, P_1^3, P_2^3, P_3^3, P_4^3.$$

$$\beta_1 \sim B_5: \alpha_1^2, \beta_1^3, P_1^3, P_2^3, P_3^3, P_4^3,$$

$$P_1 \sim V_2: \alpha_1, \beta_1, P_1^2, P_2, P_3, P_4,$$

$$P_2 \sim V_2: \alpha_1, \beta_1, P_1, P_2^2, P_3, P_4,$$

$$P_3 \sim V_2: \alpha_1, \beta_1, P_1, P_2, P_3^2, P_4,$$

$$P_4 \sim V_2: \alpha_1, \beta_1, P_1, P_2, P_3, P_4^2,$$

$$\sigma_3 \sim H,$$

$$K_4: \alpha_1^2, \beta_1^2, P_1^2, P_2^2, P_3^2, P_4^2.$$

The $(1, 2)$ correspondences between the spaces (x') and (x) can be expressed algebraically in the form

$$x_1' = x_1x_4, \quad x_2' = x_2x_3, \quad x_3' = x_1(x_3 + ax_5) + bx_3x_5,$$
$$x_4' = x_2(x_4 + cx_5) + dx_4x_5,$$
$$x_5' = (x_1 + x_2 + x_3 + x_4 + x_5)x_5.$$

From these equations the above results can be deduced algebraically. The fundamental elements have the following equations

$$\alpha_1 \equiv x_1 = x_2 = x_5 = 0;$$
$$\beta_1 \equiv x_3 = x_4 = x_5 = 0;$$
$$H\colon x_5 = 0, \quad \overline{H}\colon x_1 + x_2 + x_3 + x_4 + x_5 = 0,$$

and 2 of the points P_i are given by each of the sets of equations,

$$x_1 = x_3 = x_2 + x_4 + x_5 = x_4' = 0;$$
$$x_2 = x_4 = x_1 + x_3 + x_5 = x_3' = 0.$$

9. *Type IV.—Basis system a quadric surface* Γ_2. As in the preceding cases the image of a hyperplane V_2' is a quadric variety V_2, passing through Γ_2. The image of a plane π' is a quadric surface. A line l' has for its image a conic meeting Γ_2 in two points.

To each of the points in which an arbitrary line l' meets the image of a hyperplane V_1 there corresponds in (x) the intersection of a conic and the V_1, hence the image of V_1 is a quadric variety V_2'. To each of the points in which the image of a plane π meets an arbitrary plane of (x') there corresponds an intersection of a quadric surface with π. Hence the image of π is a quadric surface in (x'). To each of the points in which the image of a line l meets an arbitrary V_1' there corresponds an intersection of a V_2 with l.

Let H be the hyperplane which contains Γ_2. It follows from (4) that the image of O' is O and H, and that the image of any line $O'P'$ is the line OP_1P_2 and H. Any hyperplane V_1 in (x) meets H in a plane π. The lines through O meeting π have for images tangent lines to the image V_2' of V_1, hence O' is simple on V_2'. The image of V_2' is a $V_4\colon \Gamma_2^2, O$, having V_1 and H as components. The conjugate in I of V_1 is therefore a $V_2\colon \Gamma_2, O$.

Since the image of a line l' meets Γ_2 in 2 points, the image of Γ_2 is a quadric variety G_2'. Since a general line through O does not meet Γ_2, the corresponding line through O' does not meet G_2' except at O'. Hence G_2' is a quadric hypercone, vertex O'. A line through O meeting Γ_2 has for image a

generator of G_2' counted twice, and a generator of G_2' has for image a line through O meeting Γ_2. Hence the image of G_2' is the quadric hypercone G_2, vertex O and base Γ_2.

The jacobian of Σ is J_5: Γ_2^4. It consists of H counted three times and K_2: Γ_2. The image of K_2 is G_2' and L_2'.

Collecting results, we have an involution I in which

$$V_1 \sim V_2\text{: } \Gamma_2,\ O.$$
$$\Gamma_2 \sim G_2\text{: } \Gamma_2,\ O,$$
$$O \sim H,$$
$$K_3\text{: } \Gamma_2.$$

The $(1,2)$ correspondence between the spaces (x') and (x) can be expressed analytically in the form

$$x_1' = x_1x_5,\ x_2' = x_2x_5,\ x_3' = x_3x_5,\ x_4' = x_4x_5,$$
$$x_5' = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2,$$

and from these equations the above results can be deduced algebraically. The fundamental elements have the following equations:

$$\Gamma_2 \equiv x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5 = 0;$$
$$H \equiv x_5 = 0,\quad K_2 \equiv x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 = 0,$$

10. *Conclusion.* This completes the enumeration of the general types of involutions leaving invariant ∞^4 quadric varieties. The special cases in which some of the basis points are coincident or lie on a basis curve are left for later consideration.

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A Three-Dimensional Quartic Variety in Four-Space.

By Bing C. Wong.

1. In this paper we shall make use of the involutorial quartic transformation studied by Miss N. Alderton.* This transformation is effected by four hyperquadrics. To a point P is made to correspond the point P' of intersection of its polar hyperplanes with respect to the hyperquadrics. P and P' are said to be the images one of the other. By this transformation a curve of order n transforms into a curve of order $4n$, a two-dimensional variety of order n into a two-dimensional variety of $6n$, and a three-dimensional variety of order n into a three-dimensional variety of order $4n$. In particular, the image of a hyperplane S_3 is a three-dimensional quartic variety M_3^4. It is the purpose of this paper to describe a few of its most obvious features. The method is to discover the peculiarities in the hyperplane with reference to the singular manifolds of the transformation here employed and then translate them into the language of the geometry of the three-dimensional quartic variety.

2. This involutorial quartic transformation yields a two-dimensional manifold J_2^{10} of order 10 whose points have for images lines forming a three-dimensional manifold Δ_3^{15} of order 15. The lines are quadri-secants of J_2^{10} and J_2^{10} lies quadruply on Δ_3^{15}. All the three-dimensional quartic varieties which are the images of hyperplanes in four-space have in common J_2^{10}: any two intersect, in addition to J_2^{10}, in a two-dimensional manifold of order 6 which is the image of the plane common to the two corresponding hyperplanes; and any three meet in a curve of order 4 which is the image of the line common to the three corresponding hyperplanes.

3. Consider a hyperplane S_3 and let its corresponding variety be denoted by M_3^4. S_3 meets Δ_3^{15} in a surface D_2^{15} and J_2^{10} in a curve j_1^{10} lying quadruply on a D_2^{15}. Since J_2^{10} is the partial intersection of two three-dimensional quartic varieties, the curve j_1^{10} is also the partial intersection of two quartic surfaces in S_3, the residual intersection being a

* "Involutory Quartic Transformations in Space of Four Dimensions," *University of California Publications in Mathematics*, Vol. 1, pp. 345-358.

sextic curve c_1^6 having 20 points in common with it. Since c_1^6 has 7 apparent double points,* j_1^{10} has 25.† There are 20 lines meeting j_1^{10} four times ‡ and these lines are rulings of Δ_3^{15} and hence lines of the surface D_2^{15}. The image of such a line is a set of four lines concurrent at a point which is evidently a conical point on M_3^4. Hence M_3^4 has 20 conical points and is therefore of class 68.§

4. From the fact that the image of any point on j_1^{10} is a line on M_3^4 we see that M_3^4 has two systems of lines: one, the W-system, is composed of ∞ lines coming from points of j_1^{10} and the other, the T-system, composed of ∞ lines coming from the trisecants of j_1^{10}. The lines of the W-system form a two-dimensional locus which, being part of the manifold of order 60 common to M_3^4 and Δ_3^{15} including J_2^{10} four times, is of order 20. Let it be denoted by W_2^{20}. It passes through the 20 conical points of M_3^4 and, excepting those generators which meet in these points, no two generators meet.

5. Before finding the order of the locus of the lines of the T-system, let us note that the formula expressing the relation between the order n of a surface in S_3 passing through j_1^{10} k times and the order N of the corresponding variety in M_3^4 is

$$N = 6n - k \cdot 20,$$

since the transformed variety includes W_2^{20} k times.

6. The ruled surface formed by the trisecants of j_1^{10} is of order 80.‖ Let it be denoted by $\bar{T}_2^{80}$. It contains j_1^{10} as a 17-fold curve, for through each point of the curve pass 17 lines meeting the curve in two other points, contains 20 quadruple generators coincident with the quadri-secants of j_1^{10}, and also contains a nodal curve of high order. The order of the transform, T_2^{140}, which is the locus of the lines of the T-system in M_3^4, is, according to the formula above, $6 \times 80 - 20 \times 17 = 140$. This variety has 20 quadruple points coincident with the 20 conical points of M_3^4. Each of its lines meets 30 other lines of the same system, for its corresponding trisecant meets 30 other trisecants besides those which it already meets on j_1^{10}; and also

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* B. C. Wong, "On the Correspondence between Space Sextic Curves and Plane Quartics in Four-Space," *Bulletin of the American Mathematical Society*, Vol. 32, p. 156.

† Salmon, *Analytic Geometry of Three Dimensions*, Art. 345.

‡ Ibid., Art. 474.

§ Bertini, *Projektive Geometrie mehrdimensionaler Räume*, 1924, Kapitel VIII, Nr. 23, 24.

‖ Salmon, *Analytical Geometry of Three Dimensions*, Art. 471.

meets 3 lines of the W-system. Each line of the W-system meets 17 lines of T_2^{140}.

7. It is evident that any curve of order n having α points in common with J_2^{10} transforms into a curve of order $4n - \alpha$. In particular, any plane quartic curve through the 10 points in which the plane of the curve meets J_2^{10} transforms into a space sextic curve.* Let a plane containing a line τ of T_2^{140} and a line ω of W_2^{20} meet M_3^4 in those two lines and a conic λ and this conic transforms into a unicursal quintic curve in S_3 with an actual node on j_1^{10} corresponding to the two points in which τ meets λ and having two points in common with the trisecant corresponding to ω. But a plane containing two lines τ and τ' of T_2^{140} meets M_3^4 in those two lines and a conic κ which transforms into a unicursal quartic curve having two points in common with each of the two trisecants of j_1^{10} corresponding to the two lines τ and τ'.

8. We have just seen that there are conics on the variety M_3^4. In fact, there are on it four families of conics. One family, which we shall designate as the B-family, comes from the congruence of bisecants of j_1^{10}. There are in this family ∞^2 members and each member meets two lines of W_2^{20}. Since every bisecant of j_1^{10}, already meeting the surface D_2^{15} in two quadruple points on j_1^{10}, meets it again in 7 points, every conic of this family meets J_2^{10} in 7 points. From the fact that j_1^{10} has 25 apparent double points we see that through every point on M_3^4 pass 25 conics of the family.

9. Any plane meeting M_3^4 in a conic β of the B-family meets it in another conic γ which belongs to another family, which we shall designate as the C-family. There are also ∞^2 conics in this family and through every point of M_3^4 also pass 25 of them. One such conic, γ, meets J_2^{10} in only 3 points, for its companion conic β already meets the variety in 7. These 10 points are the 10 points common to J_2^{10} and the plane of the conics. Every conic γ of the C-family transforms into a unicursal quintic curve in S_3 and since γ meets its companion β in 4 points, this quintic curve has for quadrisecant the bisecant of j_1^{10} of which β is the image.

11. A conic γ' of another family, the C'-family, is obtained by intersecting M_3^4 with a plane containing a line ω of W_2^{20} and a line τ of T_2^{140}. Each conic of this family also meets J_2^{10} in 3 points. It is to this family that the conic λ of Art. 7 belongs. Properly, this family is included in the C-family. It contains ∞^1 conics.

12. If a plane contains two lines of W_2^{20} through a conical point of M_3^4,

* See B. C. Wong, "On the Correspondence, etc.," above.

it meets M_3^4 in a conic γ'' besides. Since there are 4 lines of W_2^{20} through a conical point of M_3^4 and there are 20 such conical points on M_3^4, there are in all 120 such conics arranged in 20 sets of 6 each. They are included in the C'-family. Such a conic transforms into a unicursal quintic curve with

13. There is another family, the K-family, of ∞^1 conics each of which is obtained by intersecting M_3^4 with a plane containing two lines τ and τ' of the T-system. The transform of a conic κ of this family is already discussed in Art. 7.

14. Now consider the tangent surface A_2^{40} of j_1^{10} in S_3 which is of order 40. Its image $\bar{A}_2^{200}$ in M_3^4 is of order $6 \times 40 - 2 \times 20 = 200$ and contains ∞ conics of the B-family. Each conic meets two consecutive generators of W_2^{20} and meets 36 other conics, for each generator of the developable meets 36 other generators. Any plane σ in S_3 meets A_2^{40} in a plane curve of order 40 with 10 cusps and 720 nodes. The corresponding Σ_2^6 meets $\bar{A}_2^{200}$ in a curve of order 120 having 720 actual nodes. This curve has 10 lines of W_2^{20} for tangent lines. There are in all ∞^3 such curves on M_3^4.

15. The cone K_2^9 with vertex V on j_1^{10} and elements through points of j_1^{10} is of degree 9 and has 17 double elements. The image of K_2^9 is a variety $\bar{K}_2^{34}$ of order 34 in M_3^4 with W_2^{20} discarded. This variety, $\bar{K}_2^{34}$, contains ∞ conics of the B-family all meeting the same generator v corresponding the vertex V of K_2^9 and each meeting another different generator of W_2^{20}. Seventeen of these conics are line-pairs. Hence through any line on M_3^4 which is at the same time a line of J_3^{15} can pass 17 planes which meet M_3^4 each in a conic λ and another line τ.

16. The bisecants of j_1^{10} which meet any assumed line l in S_3 form a ruled surface which has l for 25-fold line and j_1^{10} for 9-fold curve. This surface, R_2^{70}, is of order 70 and has 80 triple generators * and a nodal curve of high order. The transform of R_2^{70} is a variety of $\bar{R}_2^{240}$ of degree $6 \times 70 - 9 \times 20 = 240$ in M_3^4. On this variety, $\bar{R}_2^{240}$, is a 25-fold quartic curve λ_1^4 which is the image of the line l. Through each point of λ_1^4 pass 25 conics of the variety. Each ruling of W_2^{20} meets 9 conics and each conic meets two rulings of W_2^{20} and 28 other conics.† Corresponding to the 80 triple generators of R_2^{70} there are 80 triple lines on $\bar{R}_2^{240}$. In general, any plane section of $\bar{R}_2^{240}$ is a group of 240 points lying on a plane quartic curve, but there

* See Salmon, *Analytic Geometry of Three Dimensions*, Art. 471.

† For each generator of R_2^{70} meets 28 other generators besides those on the line l and the curve j_1^{10}.

are ∞ planes meeting $\tilde{R}_2^{240}$ in conics and 80 of them are line-pairs. Non-consecutive conics meet on a curve corresponding to the nodal curve on R_2^{70}.

17. A plane σ through the line l meets the surface R_2^{70} in a degenerate curve composed of l counted 25 times and 45 bisecants of j_1^{10}. Hence, the corresponding Σ_2^6 passes through the curve λ_1^4 on M_3^4 into which l transforms and the total intersection of Σ_2^6 and $\tilde{R}_2^{240}$ is composed of 10 lines of W_2^{20} each counted 9 times, λ_1^4 counted 25 times and 45 conics. Each of the 45 conics meets λ_1^4 once, each of the 10 lines meets 9 conics each once, and each conic meets once 28 other conics.

18. If the line l meets j_1^{10} in the point V, the ruled surface generated by a line meeting l once and j_1^{10} twice is a surface R_2^{61} of degree 61, for the cone K_2^9 of Art. 15 has to be discarded. The transform of R_2^{61} is a variety $\tilde{R}_2^{206}$. The cubic surve λ_1^3 into which l transforms is 25-fold on $\tilde{R}_2^{206}$ and each line of W_2^{20} meets 8 conics of the variety and each conic meets 21 other conics.

19. There are again 80 planes which meet $\tilde{R}_2^{206}$ in line-pairs and 17 of these pass through the line v coming from the point V on j_1^{10}. Any plane σ through l meets R_2^{61} in a degenerate curve composed of l 25 times and 36 other lines. Hence the corresponding Σ_2^6 meets $\tilde{R}_2^{206}$ in 36 conics, the cubic curve λ_1^3 counted 25 times, the line v 25 times and 9 other lines of W_2^{20} each counted 8 times.

20. Now suppose the line l is a bisecant of j_1^{10} meeting it in the points V and V'. The bisecants of j_1^{10} which meet l form a ruled surface R_2^{52} excluding the two cones with vertices at V and V'. The transform of R_2^{52} is a variety $\tilde{R}_2^{172}$ in M_3^4. Through every point on l pass 24 rulings of R_2^{52}, for l itself is a bisecant. Hence l is 24-fold on the surface and j_1^{10} is now 7-fold. The conic λ_1^2 which l has for image is 24-fold on $\tilde{R}_2^{172}$, that is, through every point on it pass 24 other conics of the variety. Each of the generators of W_2^{20} meets 7 conics of $\tilde{R}_2^{172}$ and each conic meets 14 others.

21. Among the 80 planes which meet $\tilde{R}_2^{172}$ in line-pairs there is one group of 17 all passing through the line v corresponding to the point V and another group of 17 all passing through the line v' corresponding to the point V'. Any plane σ through the line l meets R_2^{52} in a degenerate curve composed of l counted 24 times and 28 other lines. Hence the corresponding Σ_2^6 meets $\tilde{R}_2^{172}$ in λ_1^2 counted 24 times, 8 lines of W_2^{20} each counted 7 times, the lines v and v' each counted 24 times and 28 conics each counted once.

22. Let l be a trisecant of j_1^{10} meeting it in the points V, V', V''.

The ruled surface formed by the bisecants of j_1^{10} which meet l is an R_2^{43} and its transform is an $\bar{R}_2^{138}$ in M_3^4. The line l, being a trisecant, is the equivalent of three bisecants; hence it is a 22-fold line on R_2^{43}, and the line λ which is the image of l is a 22-fold line on $\bar{R}_2^{138}$, that is, through each point of λ pass 22 conics of the variety. Each ruling of W_2^{20} meets 6 conics and each conic meets 10 others.

23. Among the 80 planes which meet $\bar{R}_2^{138}$ in line-pairs there are three groups of 17 each, those of one group all passing through the line v corresponding to the point V, those of another all passing through the line v' corresponding to the point V', and those of the third all passing through the line v'' corresponding to the point V''. The Σ_2^6 corresponding to a plane σ through the line l meets $\bar{R}_2^{138}$ in the lines λ, v, v', v'' all counted 22 times, 7 rulings of W_2^{20} each counted 6 times and 21 conics each counted once.

24. If l is a quadri-secant of j_1^{10}, the locus of bisecants of j_1^{10} which meet l is a ruled surface R_2^{34} excluding the four cones with vertices at the points where l meets the curve. Its transform is a variety $\bar{R}_2^{104}$ of order 104 in M_3^4. The line l being a quadri-secant is the union of 6 bisecants, hence from every point on it 19 bisecants can be drawn to j_1^{10}. Now the transform of l is a set of four lines v, v', v'', v''' meeting in a point L which is a singular point of M_3^4 and also a singular point of $\bar{R}_2^{104}$. All the conics of this variety pass through L. Each line of W_2^{20} meets 5 conics and each conic meets 6 others.

25. All of the 80 planes which intersect $\bar{R}_2^{104}$ in line-pairs pass through L, and of these 17 pass through v, 17 through v', 17 through v'', and 17 through v'''. The variety Σ_2^6 corresponding to any plane through l meets $\bar{R}_2^{104}$ in the lines v, v', v'', v''' each counted 19 times, 6 lines of W_2^{20} each counted 5 times, and 15 conics each once.

26. It remains to mention that any quartic surface F_2^4 in S_3 which contains j_1^{10} transforms into a two-dimensional variety $\bar{F}_2^4$ also of order 4 in M_3^4 as is seen from the formula $N = 6n - k \cdot 20$ for $n = 4$, $k = 1$. $\bar{F}_2^4$ is of the same nature as F_2^4. The fact that F_2^4 contains j_1^{10} shows that it is the intersection of S_3 and a three-dimensional quartic variety $\bar{M}_3^4$ which is the image of a hyperplane $\bar{S}_3$, and that $\bar{F}_2^4$ is the intersection of $\bar{S}_3$ with M_3^4. There is one quartic surface, I_2^4, in S_3 which is its own image and that is the surface in which S_3 meets its own transform M_3^4.

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Compound Singularities of the Rational Plane Quartic Curve.

By J. H. Neelley.

1. *Introduction.* This paper is concerned with those types of the rational plane quartic curve which have two or more coincident singularities, known as compound singularities.* In the first part certain theorems are obtained and the invariant conditions are found for the existence of such types. In the second part we introduce some special quartic surfaces whose plane sections include all possible types of rational quartics with compound singularities.

We represent the rational plane quartic $R^4{}_2$ by the three binary forms

$$x_i = (\alpha_i t)^4 \qquad (i = 0, 1, 2)$$

and then deduce the pencil of apolar quartics

$$(\beta t)^4 + \lambda(\gamma t)^4$$

The combinants of this pencil involve determinants of the type $\begin{vmatrix} \beta_i & \beta_j \\ \gamma_i & \gamma_j \end{vmatrix}$ which are usually designated by p_{ij}. The invariant conditions for compound singularities are expressed in terms of these combinants.

2. *The Ramphoid Cusp.*† The ramphoid cusp or cusp of the second kind has been called by Cayley a node-cusp. To find the necessary and sufficient conditions for its existence, let the cusp parameter be $t = 0$ and the parameter value of an inflectional point $t = \infty$. Then taking as a reference scheme the tangents to the curve at the cusp and at the point of inflection and the line joining these points, the fundamental quartics ‡ are

$$(1) \qquad (\beta t)^4 \equiv 12bt^2 + 12ct, \quad (\gamma t)^4 \equiv 4dt^4 + 4et^3$$

Hence $p_{01} = p_{04} = p_{14} = p_{23} = p_{24} = p_{34} = 0$

$$p_{02} = 8bd,\ p_{03} = 12cd,\ p_{12} = 2be,\ p_{13} = 3ce.$$

* Cayley, *Quarterly Journal*, Vol. 7, p. 212.

† Salmon, *Higher Plane Curves*, Chapter VI; F. Klein, *Projective Geometry*, p. 108.

‡ W. Stahl, "Ueber die rationalen ebenen Curve vierter Ordnung," *Crelle's Journal für Mathematik*, Vol. 101, p. 305.

The sextic giving the parameter values of the inflectional points * becomes

$$(st)^6 \equiv 2bdt^5 + (3cd + be)t^4 + 2cet^3 = 0. \tag{2}$$

So three points of inflection coincide at the ramphoid cusp. We note that the other two intersections of the quartic and the conic on the inflectional points † have their parameter values given by

$$(2cd - be)t^2 - cet = 0 \tag{3}$$

which shows that the conic on the points of inflection cuts out four coincident points of R_2^4 at a ramphoid cusp.

Upon consideration of the five fundamental invariants ‡ of R_2^4 we find that

$$I'_2 - 4I_2 = 0, \quad I_4 = 0, \quad I_6 = 0. \tag{4}$$

The first of these invariants vanishes when the curve has three double lines on a point; I_4 is the undulation invariant and I_6 the cusp invariant.§ So the curve with a ramphoid cusp satisfies both the cusp and undulation conditions. This follows from the fact that two points of inflection coincide to form either a cusp or an undulation as the discriminant of the sextic giving the parameters of the inflectional points involves both cusp and undulation invariants.‖ This discriminant, being of degree 10 in the p's, can only differ from the product of these two invariants by a constant factor and we find that

$$2^6\Delta = I_4 I_6. \tag{5}$$

The curve may have cusps without affecting the fundamental involution but undulations do affect this involution. The ambiguity on this point which is evident in most references is removed by the following. When the curve has one undulation the fundamental sextic becomes the sixth power of the undulation. And as line sections of the curve are the second polars of the

* A. Brill, "Ueber rationale Curve vierter Ordnung," *Mathematische Annalen*, Vol. 12, p. 104; J. E. Rowe, "Bicombinants of Rational Plane Quartic, etc.", *Transactions of the American Mathematical Society*, Vol. 13.

† Stahl, "Rationale ebene Curven vierter Ordnung," *Crelle's Journal für Mathematik*, Vol. 101, p. 307.

‡ The Morley invariants as given by Winger: *American Journal of Mathematics*, Vol. 36, p. 55, and Rowe, *Transactions of the American Mathematical Society*, Vol. 12, p. 304.

§ Rowe, *Transactions of the American Mathematical Society*, Vol. 12, p. 304.

‖ A. Brill, "Ueber rationale Curve vierter Ordnung," *Mathematische Annalen*, Vol. 12, p. 112; Stahl, "Rationale ebene Curven vierter Ordnung," *Crelle's Journal für Mathematik*, Vol. 104, p. 319.

sextic and three independent ones recover the curve this type curve cannot be recovered as only the undulation line can be gotten from the sextic. Then if the curve has two undulations the sextic is not unique but becomes a pencil of sextics which includes the sixth powers of the undulation parameters as members. And again the curve cannot be recovered from the sextic as all second polars are lines of a pencil which includes the two undulation lines. If the curve has three undulations there is a net of sextics which includes the sixth powers of the three undulations as members. Any member of the net which is a linear combination of each of these sixth powers will have a net of second polars which recovers the curve. For instance, if the undulation parameters are 0, ∞, $+1$, the net is

$$\lambda t^6 + \mu T^6 + \nu(t - T)^6 = 0$$

and if λ, μ and ν are not zero the second polars will recover the curve.

The curve may have an undulation or a cusp in addition to a ramphoid cusp, but requiring the curve to have two ramphoid cusps forces it to degenerate into two double contact conics.

The condition that a binary sextic have a triple root is that Salmon's three invariants A, B and C must satisfy the three relations*

$$(6) \qquad A^2 - 100B = 0, \quad 4AB - 5C = 0, \quad AC - 80B^2 = 0.$$

We shall now show the equivalence of conditions (4) and (6). Computing all invariants involved and using the general form of the flex sextic with one root 0 and one ∞ we have

$$I_2 = p_{03}p_{23} - p_{02}p_{24} + p_{12}p_{14} + p_{12}p_{23} - p^2_{13}.$$

$$I'_2 = 8p_{04}p_{13} - 12p_{03}p_{23} - 12p_{12}p_{14} + 36p_{12}p_{23} - p^2_{04} - 4p^2_{13}.$$

$$I_4 = p^4_{04} - 16p^3_{04}p_{13} - 72p_{02}p^2_{04}p_{24} + 96p_{03}p^2_{04}p_{23} - 576p^2_{04}p_{12}p_{23} + 96p^2_{04}p_{12}p_{14} + 1296p^2_{02}p^2_{24}.$$

$$I_6 = 36p_{04}p^2_{12}p_{13}p^2_{23} - 42p_{04}p^3_{12}p_{23}p_{24} - 42p^2_{02}p_{14}p^3_{23} + 6p_{02}p^2_{12}p^2_{23}p_{24} - 9p^2_{02}p^2_{13}p^2_{24} + 12p^2_{02}p_{03}p_{23}p^2_{24} - 24p_{03}p^2_{12}p^3_{23} - 24p^3_{12}p_{14}p^2_{23} + 16p^3_{12}p^3_{23} + 12p^2_{02}p_{12}p_{14}p^2_{24} + 48p^2_{02}p_{12}p_{23}p^2_{24} - 16p^3_{02}p^3_{24} + 27p^2_{02}p^4_{23} + 27p^4_{12}p^2_{24}.$$

$$A = -(1/40)(p^2_{04} - 8p_{04}p_{13} + 60p_{02}p_{24} - 48p_{03}p_{23} - 48p_{12}p_{14} - 96p_{12}p_{23} + 64p^2_{13}).$$

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* Salmon, *Modern Higher Algebra*, 4th edition, p. 263.

$$B = (1/2^4 \cdot 10^4)\ (p^4_{04} + 10000p^2_{02}p^2_{24} - 4032p^2_{04}p_{12}p_{23} - 5120p_{02}p_{04}p^2_{23}$$
$$- 5120p_{04}p^2_{12}p_{24} - 12800p_{02}p_{12}p_{23}p_{24} - 200p_{02}p^2_{04}p_{24}$$
$$+ 8704p_{02}p^2_{13}p_{24} - 16p^3_{04}p_{13} - 128p^2_{04}p^2_{13} - 1024p_{04}p^3_{13}$$
$$+ 4096p^4_{13} + 224p^2_{04}p_{12}p_{14} + 2048p_{04}p_{12}p_{13}p_{14} + 14336p_{03}p_{12}p^2_{23}$$
$$+ 224p_{03}p^2_{04}p_{23} + 2048p^2_{03}p_{13}p_{24} + 14336p^2_{12}p_{13}p_{24} + 1024p^2_{03}p^2_{23}$$
$$+ 4096p^2_{12}p^2_{23} + 1024p^2_{12}p^2_{14} - 6144p_{03}p^2_{13}p_{23} - 6144p_{12}p^2_{13}p_{14}$$
$$- 12288p_{12}p^2_{13}p_{23}).$$

$$C = (1/2^9 \cdot 5^6)\ (-p^6_{04} + 1000000p^3_{02}p^3_{24} - 262144p^6_{13}$$
$$- 67200p_{02}p^2_{04}p_{12}p_{23}p_{24} - 480000p^2_{02}p_{03}p_{23}p^2_{24}$$
$$- 480000p_{02}p_{04}p^2_{12}p^2_{24} - 1920000p^2_{02}p_{12}p_{23}p^2_{24} - 30000p^2_{02}p^2_{04}p^2_{24}$$
$$- 1920000p^2_{02}p^2_{13}p^2_{24} + 84480p^3_{04}p^2_{12}p_{24} - 521968p^2_{04}p^2_{12}p_{14}p_{23}$$
$$+ 1915392p^2_{04}p^2_{12}p^2_{23} - 122880p^2_{02}p_{14}p^3_{23} + 24p^5_{04}p_{13}$$
$$+ 96p^4_{04}p^2_{13} - 6656p^3_{04}p^3_{13} + 6144p^2_{04}p^4_{13} + 98304p_{04}p^5_{13}$$
$$- 49152p^2_{02}p^4_{14} - 1382400p^2_{02}p^4_{23} + 84480p_{02}p_{03}p^2_{04}p_{23}p_{24}$$
$$- 511968p^3_{03}p_{12}p^2_{24} - 122880p_{03}p^3_{12}p^2_{24} - 99072p^3_{04}p_{12}p_{13}p_{23}$$
$$- 215040p_{02}p^2_{12}p^2_{14}p_{24} - 215040p_{02}p^2_{03}p^2_{23}p_{24}$$
$$- 1536000p_{02}p^2_{12}p^2_{23}p_{24} - 336p^4_{04}p_{12}p_{14} + 9216p^3_{04}p_{12}p_{13}p_{14}$$
$$+ 27648p^2_{04}p_{12}p^2_{13}p_{14} - 270336p_{04}p_{12}p^3_{13}p_{14} - 10752p^4_{04}p_{12}p_{23}$$
$$+ 473088p^2_{04}p_{12}p^2_{13}p_{23} - 1425408p_{04}p_{12}p^3_{13}p_{23}$$
$$+ 921600p_{02}p_{04}p^2_{13}p^2_{23} - 2260992p_{02}p^3_{13}p^2_{23} + 921600p_{04}p^2_{12}p^2_{13}p_{24}$$
$$+ 2899968p_{02}p_{12}p^2_{13}p_{23}p_{24} + 208896p_{04}p^2_{12}p_{13}p^2_{14}$$
$$+ 1155072p^2_{03}p_{12}p^3_{23} + 2310144p_{03}p^2_{12}p^3_{23} + 300p_{02}p^4_{04}p_{24}$$
$$- 49152p^2_{03}p^2_{04}p^2_{23} - 1382400p^4_{12}p^2_{24} + 9216p^3_{03}p_{04}p_{14}p_{24}$$
$$+ 27648p^3_{03}p^2_{14}p_{23} + 208896p^3_{03}p_{14}p^2_{23} + 1155072p^3_{12}p^2_{14}p_{23}$$
$$+ 2310144p^3_{12}p_{13}p_{23}p_{24} - 32768p^3_{12}p^3_{14} - 32768p^3_{03}p^3_{23}$$
$$- 262144p^3_{12}p^3_{23} - 336p_{03}p^4_{04}p_{23} - 270336p^2_{03}p^3_{13}p_{24}$$
$$- 2260992p^2_{12}p^3_{13}p_{24} - 319488p^2_{12}p^2_{13}p^2_{14} - 319488p^2_{03}p^2_{13}p^2_{23}$$
$$- 1277952p^2_{12}p^2_{13}p^2_{23} + 589824p_{03}p^4_{13}p_{23} + 589824p_{12}p^4_{13}p_{14}$$
$$+ 1179648p_{12}p^4_{13}p_{23}).$$

From these we obtain the relations

$$(7) \qquad (I'_2 - 4I_2)^2 - I_4 \equiv 320(A^2 - 100B)$$

$$(8) \qquad 3072I_6 - 192I_2I_4 + (I'_2 - 4I_2)^2\ (I'_2 - 20I_2) \equiv -2^4 \cdot 10^4(4AB - 5C)$$

$$(9)\qquad 10240I_6(60I_2+I'_2)+5I_4{}^2+I_4(10640I_2{}^2+440I_2I'_2-7I'_2{}^2)$$
$$-2(I'_2-4I_2)^2(560I_2{}^2+120I_2I'_2-I'_2{}^2)\equiv -2^6\cdot 10^6(AC-80B^2)$$

So if equations (4) are satisfied the conditions that the flex sextic have a triple root are fulfilled. Thus the conditions (4) are sufficient to make three flexes coincident. To show that they are necessary, suppose the conditions (6) satisfied, whence

$$(10)\qquad I_4=(I'_2-4I_2)^2$$

$$(11)\qquad 3072I_6=(I'_2-4I_2)^2(212I_2-I'_2)$$

$$(12)\qquad (I'_2-4I_2)^2(260I_2-I'_2)\ (60I_2+I'_2)=0$$

Since $(st)^6$ has a triple root its discriminant vanishes, whence

$$(13)\qquad (1/2^6)I_4I_6=0.$$

Therefore either $I_4=0$ or $I_6=0$. If $I_4=0$ then $I'_2-4I_2=0$ and then $I_6=0$ from (10) and (11). If I_6 vanishes there are two cases. Either I'_2-4I_2 or $212I_2-I'_2$ vanishes. If $I'_2-4I_2=0$, I_4 vanishes and conditions (4) are satisfied. But if I'_2-4I_2 does not vanish we have $212I_2-I'_2=0$ and either $260I_2-I'_2=0$ or $60I_2+I'_2=0$. In any case all second order invariants of the curve vanish and consequently all invariants vanish. This form of $R_2{}^4$ has been discussed by Winger,* and avoiding this special type of the curve we have the theorem: *The necessary and sufficient condition that the rational plane quartic have a ramphoid cusp is that three points of inflection coincide. This requires that the three invariants I'_2—$4I_2$, I_4 and I_6 vanish simultaneously.*

3. *The Tacnode.* The tacnode has been considered as the result of two coincident nodes and as the nodes are associated with the catalectic sets of the fundamental pencil the tacnode condition has been given as the coincidence of two of the catalectic sets.† If the fundamental pencil associated with $R_2{}^4$ is

$$(\beta t)^4+\lambda(\gamma t)^4$$

the catalecticant is

$$(1)\qquad \begin{vmatrix} \beta_0+\lambda\gamma_0 & \beta_1+\lambda\gamma_1 & \beta_2+\lambda\gamma_2 \\ \beta_1+\lambda\gamma_1 & \beta_2+\lambda\gamma_2 & \beta_3+\lambda\gamma_3 \\ \beta_2+\lambda\gamma_2 & \beta_3+\lambda\gamma_3 & \beta_4+\lambda\gamma_4 \end{vmatrix}=0$$

* *American Journal of Mathematics*, Vol. 36, p. 56.

† Rowe, *Transactions of the American Mathematical Society*, Vol. 12, p. 304.

and its discriminant is

$$(2) \qquad -(1/2^6)\{I_2I_4 - I_2(I'_2 - 4I_2)^2 - 64I_6\}$$

which is taken as the tacnodal invariant. It is obvious that the ramphoid cusp is one type of the tacnode as its conditions make (2) vanish.

Consider R_2^4 with a tacnode and an acnode. Call the contacts of the two tangents to the curve from the tacnode 0 and ∞ and the pencil of lines on the tacnode cuts out the involution $t + T = 0$. Naming one tacnodal parameter 1 the other is -1. Then using the real double line and the joints of its contacts t_1 and t_2 to the tacnode as reference lines we have R_2^4 given by

$$(3) \qquad \begin{aligned} x_0 &= (t^2 - 1)(t^2 - t_2^2) \\ x_1 &= (t^2 - 1)(t^2 - t_1^2) \\ x_2 &= (t - t_1)^2(t - t_2)^2 \end{aligned}$$

and

$$(4) \qquad \begin{aligned} (\beta t)^4 &\equiv 4t^3 - 4t_1t_2t \\ (\gamma t)^4 &\equiv t_1t_2(t_1 + t_2)t^4 - 2(1 + t_1^2 + t_2^2 + t_1^2t_2^2 + 4t_1t_2)t^3 \\ &\qquad + 6t_1t_2(t_1 + t_2)t^2 + t_1t_2(t_1 + t_2) \end{aligned}$$

The parameters of the acnode are

$$(5) \qquad t = \pm\,(-t_1t_2)^{1/2}$$

Now the line on the tacnode which with the tacnodal line forms a pair apolar to x_0 and x_1 of (3) cuts out the quartic

$$(6) \qquad (t_1^2 + t_2^2 - 2)t^4 - 2(t_1^2t_2^2 - 1)t^2 + (2t_1^2t_2^2 - t_1^2 - t_2^2)$$

If this line is on the acnode $t_1 = 1/t_2$ and the acnodal parameters are $\pm i$. Then the curve is self-projective. We show this by observing that Salmon's invariant M, the equivalent of the Morley invariant I_9, vanishes.

$$(7) \qquad M^2 = A(AE - I^2)^2 - 8D(I^3 - 9AEI + 54DE)$$

where $E = 0$ is the tacnodal condition, so

$$(8) \qquad M^2 = I^3(AI - 8D) \text{ and } AI - 8D = 0.$$

However $I \neq 0$. This condition (8) for a self-projective R_2^4 with a tacnode in the Morley invariants is

$$(9) \qquad I_2'^3 + 68I_2'^2I_2 + 432I'_2I_2^2 - 2880I_2^3 - I'_2I_4 - 12I_2I_4 - 1024I_6 = 0.$$

If the curve has a tacnode and a crunode the reference scheme gives the crunodal parameters as $\pm(t_1t_2)^{1/2}$. Now to make this curve self-projective

the crunode must lie on the line (6), but in that case the crunode coincides with the tacnode and we have an oscnode.* This type of curve will be discussed in the next section of this paper.

We wish to establish the theorem that a tacnode is the result of two coincident double lines of the curve. Call two inflectional parameters $t = 0$ and $t = \infty$ and use the inflectional lines and their join as a triangle of reference. Then R_2^4 is

$$\begin{aligned} x_0 &= t^4/e + t^3/d \\ x_1 &= \quad t^3/d + t^2/c + t/b \\ x_2 &= \quad\quad\quad t/b + 1/a \end{aligned} \tag{10}$$

and

$$(\beta t)^4 \equiv at^4 + 4bt^3 + 6ct^2, \quad (\gamma t)^4 \equiv 6ct^2 + 4dt + e. \tag{11}$$

A double line of the curve is of the form

$$(t^2 + 2\lambda t + \mu)^2 \tag{12}$$

and the quartic (12) must be apolar to the two base quartics (11). So

$$\begin{aligned} a\mu^2 - 4b\lambda\mu + 2c\mu + 4c\lambda^2 &= 0 \\ 4c\lambda^2 - 4d\lambda + 2c\mu + e &= 0. \end{aligned} \tag{13}$$

Eliminating λ from (13) we have

$$\begin{aligned} a^2c\mu^4 - 4b(ad - 2bc)\mu^3 + 2(2ad^2 + 2b^2e - 8bcd - ace)\mu^2 \\ - 4d(be - 2cd)\mu + ce^2 = 0 \end{aligned} \tag{14}$$

which gives the four values of μ which, together with the four corresponding values of λ, make (12) represent the four double lines of R_2^4. The discriminant of equation (14) is

$$(1/4)(ad^2 - b^2e)^2 I'_6. \tag{15}$$

Where I'_6 is the tacnodal invariant. So we have two possible conditions for (14) to have a double root. But $ad^2 - b^2e = 0$ does not give a double root for λ and so (12) does not represent two equal forms under this condition, as we have four distinct double lines. Thus a tacnode is the only condition upon which we have two coincident double lines of the curve. If μ were eliminated from equations (13) we would come to the same conclusion from a study of the resulting quartic in λ. Hence the theorem: *The necessary and sufficient condition for a plane rational quartic to have a tacnode is that two double lines be coincident.*† This theorem shows that the tacnode is a

* Salmon, *Higher Plane Curves*, 2nd edition, p. 207.

† Wieleitner, *Theorie der algebraischen, höhern Kurven.*

self-dual compound singularity, two coincident nodes or two coincident double lines being the necessary and sufficient condition for such.

We may distinguish between the tacnode which is an *embrassement* and the one which is an *opposition* by consideration of the quartic (14). This quartic has a double root when the curve has a tacnode and the other two roots form the coefficients of the remaining two distinct double lines. In the case of the *embrassement* the distinct double lines are real and so the distinct roots of (13) must be real. But for the *opposition* the remaining double lines are imaginary and so the distinct roots of the quartic must be imaginary. So we have the *embrassement* or *opposition* according as

$$2Hg_2 - 3\alpha g_3 \lesseqgtr 0 \tag{16}$$

where α is the leading coefficient of (13) and H, g_2 and g_3 are the regular invariants of the binary quartic.*

4. *The Oscnode.* The three nodes of the rational plane quartic may coincide as consecutive points giving rise, not to a triple point, but to an oscnode which is an osculation of two branches of the curve. We shall now show that this compound singularity is due to three coincident double lines. With a reference scheme similar to that of the previous section the catalectic cubic is

$$\begin{aligned} &c^3\lambda + (3c^3 + b^2e - 2bcd - ace)\lambda^2 \\ &\qquad + (3c^3 + ad^2 - 2bcd - ace)\lambda + c^3 = 0 \end{aligned} \tag{1}$$

and the quartic giving the coefficient of the double lines is

$$\begin{aligned} &16ac^2\lambda^4 - 32(acd - bc^2)\lambda^3 + 8(ace + 2ad^2 - 4bcd)\lambda^2 \\ &\qquad - 8(ade - bce - 2c^2d)\lambda + ae^2 - 4c^2e = 0. \end{aligned} \tag{2}$$

We have an oscnode when (1) is a cube and the condition is the identical vanishing of its Hessian. That is

$$\frac{3c^3}{3c^3 + b^2e - 2bcd - ace} = \frac{3c^3 + b^2e - 2bcd - ace}{3c^3 + ad^2 - 2bcd - ace} = \frac{3c^3 + ad^2 - 2bcd - ac}{3c^3}$$

Equating these in pairs we have the condition that (1) be a cube. That is

$$A = B = C = 0 \tag{3}$$

where

* Burnside and Panton, *Theory of Equations*, Vol. 1, Ex. 31, p. 153.

$$A \equiv a^2c^2e^2 + a^2d^4 - 2a^2cd^2e + 6ac^3d^2 - 3ac^4e - 4abcd^3 + 4abc^2de + 4b^2c^2d^2 - 3b^2c^3e - 6bc^4d.$$

$$B \equiv a^2cd^2e - a^2c^2e^2 - ab^2d^2e + ab^2ce^2 + 6ac^4e - 4abc^2de + 2abcd^3 + 12bc^4d - 3ac^3d^2 - 3b^2c^3e - 4b^2c^2d^2 + 2b^3cde.$$

$$C \equiv a^2c^2e^2 - 3ac^3d^2 - 3ac^4e - 2ab^2ce^2 + 4abc^2de - 6bc^4d + 6b^2c^3e + 4b^2c^2d^2 - 4b^3cde + b^4e^2.$$

Then the conditions that equation (2) have a triple root are that its invariants g_2 and g_3 vanish. Computing g_2 we find that it is $(3/2^6)A$. Now using the fact that the Hessian of cubic (1) has the same discriminant as the cubic we have

$$4AC - B^2 = KI'_6 \tag{4}$$

where I'_6 is the tacnodal invariant. But the discriminant of the quartic (2) is

$$g_2^3 - 27g_3^2 \equiv 2^{10}c^6I'_6$$

whence

$$g_2^3 - 27g_3^2 \equiv Kc^6(4AC - B^2). \tag{5}$$

And relation (5) shows that three coincident nodes necessitates three coincident double lines. Then the converse is true when we neglect the factor c. This may be done as $c = 0$ gives a curve with two distinct cusps and so cannot be a type of oscnode.* So we have the theorem: *The necessary and sufficient condition for a rational plane quartic curve to have an oscnode is the coincidence of three double lines of the curve.*†

5. *The Oscnodal Conditions.* Considering an oscnode as formed by sending the third node to the point of a tacnode on the curve and using a reference scheme similar to that proposed for the tacnode we find that the contacts of the real double line are in the relation $t_2 = \pm 1/t_1$, where the sign depends upon the third node being an acnode or a crunode. In both cases the curve may be represented by

$$\begin{aligned} x_0 &= t^4 - (1 + 1/t_1^2)t^2 + 1/t_1^2 \\ x_1 &= t^4 - (1 + t_1^2)t^2 + t_1^2 \\ x_2 &= t^4 - 2(t_1 - 1/t_1)t^3 + (t_1^2 - 4 + 1/t_1^2)t^2 + 2(t_1 - 1/t_1)t + 1 \end{aligned} \tag{1}$$

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* If 6σ and $6\sigma'$ are the coefficients of t^3 in the members of the fundamental pencil we do not have two cusps if either σ or σ' vanishes but the curve has distinct double lines.

† Wieleitner, *l. c.*

and

$$(2) \qquad \begin{aligned} (\beta t)^4 &\equiv t^3 + t \\ (\gamma t)^4 &\equiv t^4 + 2(t_1 - 1/t_1)t^3 + 6t^2 + 1. \end{aligned}$$

The catalecticant is $\lambda^3 = 0$ and consideration of the Morley invariants shows that together with the tacnode invariant there is an

$$\bar{I}_4 \equiv (I'_2 - 4I_2)(I'_2 + 60I_2) - I_4$$

which vanishes. When this $\bar{I}_4$ is calculated for a curve with a tacnode, it vanishes if the third node is at the tacnode. But $\bar{I}_4 \equiv 0$ for the ramphoid cusp so to distinguish between the two types of curve it is necessary that $I_6 \neq 0$. Hence the theorem: *The necessary and sufficient conditions for an oscnode are*

$$I'_6 = 0, \quad \bar{I}_4 = 0, \quad I_6 \neq 0.$$

If there is a cusp at the oscnode we have the same curve which would arise if the ramphoid cusp type had the third node coincident with the cusp. In addition to the ramphoid cusp invariants the type requires that each of the self-apolar members of the fundamental pencil be apolar to the Hessian of the other.

Using for reference lines the line on the oscnode which with the oscnodal line forms a pair apolar to x_0 and x_1 of (1), the oscnodal line and the other double line, the curve is given by

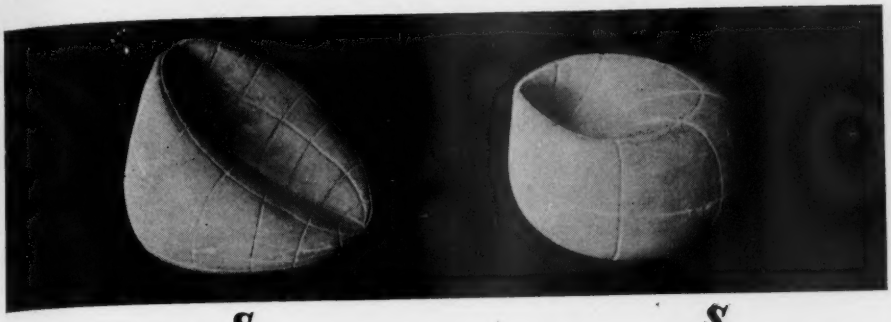
$$(3) \qquad \begin{aligned} x_0 &= t^4 - 1 \\ x_1 &= t^4 - 2t^2 + 1 \\ x_2 &= \{t^2 - (t_1 - 1/t_1)t - 1\}^2. \end{aligned}$$

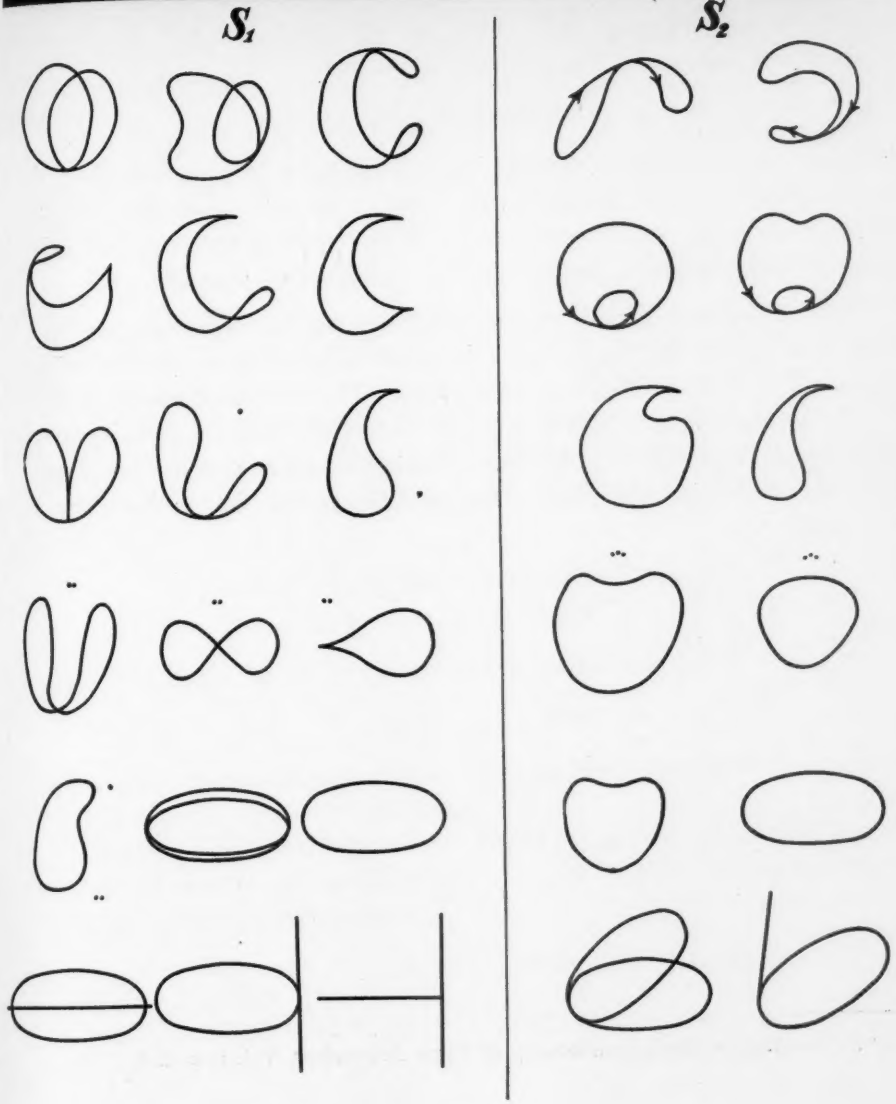
This representation shows that the curve is reflected into itself by $(1, 0, 0)$ and x_0. That this type is self-projective is easily verified by the vanishing of the Morley invariant I_9.

$$(4) \qquad I_9^2 = A(AE - I^2) - 8D(I^3 - 9AEI + 54DE)$$

where E is the tacnodal invariant and I is the oscnodal invariant introduced above.

Attention is called, however, to the fact that the invariant form $AI - 8D$ does not vanish for an oscnode. Remembering that it does vanish for the self-projective curve with a tacnode and that the invariant I does not, we have a method for distinguishing between the oscnodal and the self-projective curve with a tacnode.




S_1
S_2

6. *The Tacnodal and Oscnodal Surfaces.* It is well known that all forms of the rational plane quartic with simple singularities are, to within projection, plane sections of the Steiner Surface. This surface has three double lines which give the nodes of the plane sections. Now curves with tacnodes have two coincident nodes, considering this fact and using Salmon's suggestions for modified forms of the equation of the Steiner Surface,* we have constructed a tacnodal surface S_1, which has the equation

$$x_0^4 + x_1^2 x_2^2 + x_0^2 x_1 x_3 = 0.$$

The second surface S_2 we present has one double line and the surface osculates itself along this line and has the equation

$$(x_1^2 - x_0 x_2)^2 - x_0^3 x_3 = 0.$$

All plane sections of these surfaces are rational quartics with tacnodes and oscnodes and, to within projection, all forms of the curve with these singularities may be cut from these surfaces. These surfaces and types of their plane section are given in the lower part of the figure on page 399.

The curves with isolated tacnodes and oscnodes have these singularities represented by two and three dots respectively and such belong to the curve immediately below or to the left of the dots. The last five drawings under S_1 are degenerate forms which include the conic of the tropes and the squares of the two double lines. The last three drawings under S_2 are also degenerate forms being the trope conic, two osculating conics and the double line tangent to a conic.

* Salmon-Rogers, *Analytic Geometry of Three Dimensions*, Vol. 2, p. 214.

Pencils of Conics in the Galois Fields of Order 2^n.

By Alan D. Campbell.

We define a conic in a Galois field of order 2^n [which field we write more briefly $GF(2^n)$] as the locus of all points whose coordinates x, y, z satisfy an equation of the form

$$ax^2 + by^2 + cz^2 + fyz + gzx + hxy = 0, \tag{1}$$

whose coefficients and variables represent numbers in this $GF(2^n)$. The discriminant of (1) is

$$\Delta \equiv fgh + af^2 + bg^2 + ch^2. \tag{2}$$

A pencil of conics then has the equation

$$\lambda C_1 + \mu C_2 = 0, \tag{3}$$

where $C_1 = 0$ and $C_2 = 0$ are two distinct conics. In this paper we derive the classes, or sets of classes, of projectively equivalent pencils and we give a typical pencil for each class, or set of classes. If there is an arbitrary coefficient in the typical pencil we say this pencil represents a set of classes, whenever different values of this coefficient may give nonequivalent pencils and so represent distinct classes. The discriminant Δ of the general conic of a pencil we call the discriminant of the pencil. If we subject (3) to the transformation

$$x = a_1x' + b_1y' + c_1z', \quad y = a_2x' + b_2y' + c_2z', \quad z = a_3x' + b_3y' + c_3z', \tag{4}$$

with determinant $|\, a_1,\ b_2,\ c_3 \,| \neq 0$, we get for the new pencil

$$\Delta' \equiv |\, a_1,\ b_2,\ c_3 \,|^2\, \Delta.$$

(After such a transformation as this we drop the primes from the variables, without explicit mention of this fact).

We note that in a $GF(2^n)$ every number is a perfect square with just one square root, also $(\alpha x + \beta y + \gamma z)^2 = \alpha^2x^2 + \beta^2y^2 + \gamma^2z^2$. In any $GF(2^n)$ we have the following typical conics

$$x^2 + yz = 0, \tag{5}$$

$$xy = 0, \tag{6}$$

(7) $$x^2 + y^2 + \alpha xy = 0,$$

where $z^2 + \alpha z + 1 = 0$, $z = x/y$, is irreducible in this $GF(2^n)$*,

(8) $$x^2 = 0.$$

We first divide the classes of pencils of conics into the following sets:

Set I. Each pencil of this set contains at least one double line.

Set II. Each pencil has no double line, but has at least one real line pair.

Set III. Each pencil has no double line and no real line pair, but has at least one conjugate imaginary line pair.

Set IV. Each pencil has no degenerate conics of any description.

Set I. We put (3) in a form (A) where $C_1 \equiv x^2$ and $C_2 = 0$ is of the form (1) with $a = 0$. If $f = g = h = 0$, $b \neq 0$ in (A), we put

$$x = x', \quad y = (1/b^{\frac{1}{2}})y' + (c^{\frac{1}{2}}/b^{\frac{1}{2}})z', \quad z = z',$$

and we get the following class with a typical pencil and its discriminant

(9) Class 1, $$\lambda x^2 + \mu y^2 = 0, \quad \Delta \equiv 0.$$

If $h \neq 0$, $f = 0$ in (A),† we put

$$x = x', \quad y = (1/h)y' + (g/h)z', \quad z = z',$$

and we get a pencil (A') with $C_1' \equiv x^2$ and $a' = f' = g' = 0$, $h' = 1$ in C_2'. If $c' \neq 0$ in (A'), we obtain by an obvious transformation

(10) Class 2, $$\lambda x^2 + \mu(z^2 + xy) = 0, \quad \Delta \equiv \mu^3.$$

If $c' = 0$, $b' \neq 0$ in (A'), we have

(11) Class 3, $$\lambda x^2 + \mu(y^2 + xy) = 0, \quad \Delta \equiv 0.$$

The pencil (11) has only one double line and so is not equivalent to (9).

If $b' = c' = 0$ in (A'), we have

(12) Class 4, $$\lambda x^2 + \mu xy = 0, \quad \Delta \equiv 0.$$

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* See L. E. Dickson, "Linear Groups," p. 199.

† If $h = f = 0$, $g \neq 0$, we put $x = x'$, $y = z'$, $z = y'$ and we have again the case $h \neq 0$, $f = 0$.

The pencil (12) has just one double line and so is not equivalent to (9). Any transformation (4) that is to send (11) into (12) must have $b_1 = c_1 = c_2 = 0$, $b_2 c_3 \neq 0$. But then (4) sends C_2 of (11) into $a_2^2 x'^2 + b_2^2 y'^2 + \cdots$ which has a term in y'^2, whereas (12) has no such term.

If $f \neq 0$ in (A), we can reduce this pencil to a form (A''), with $C_1' \equiv x^2$ and C_2' having $a' = h' = g' = 0$, $f' = 1$. If $C_2' = 0$ is reducible to (7) we get

(13) Class 5, $\lambda x^2 + \mu(y^2 + z^2 + \alpha yz) = 0, \quad \Delta \equiv \alpha^2 \lambda \mu^2.$

If $C_2' = 0$ in (A'') is reducible to (6), we get

(14) Class 6, $\lambda x^2 + \mu yz = 0, \quad \Delta \equiv \lambda \mu^2.$

The pencil (14) has a real line pair and so is nonequivalent to (13).

Set II. We put (3) in a form (B) where $C_1 \equiv xy$ and C_2 has $h = 0$. If $cg \neq 0$ in (B)* and $ax^2 + cz^2 + gzx$ is factorable, we can reduce (B) to a form (B') with $C_1' \equiv xy$ and $a' = 0$ $c'g' \neq 0$. If in (B') $b'y^2 + f'yz + c'z^2$ is factorable and $f' \neq 0$, we can reduce (B') to a form (B'') with $C_1'' \equiv xy$ and $c''f''g'' \neq 0$, $a'' = b'' = h'' = 0$, and we obtain

(15) Class 7, $\lambda xy + \mu(z^2 + yz + zx) = 0, \quad \Delta \equiv \lambda\mu(\lambda + \mu).$

If in (B') $f' \neq 0$ and the terms in y and z alone are not factorable, we obtain

(16) Class 8, $\lambda xy + \mu(z^2 + y^2 + \alpha yz + zx) = 0,$
$\Delta \equiv \mu(\lambda^2 + \mu^2 + \alpha\lambda\mu),$

where α makes the quadratic factor in Δ irreducible.

If in (B') $f' = 0$, we have

(17) Class 9, $\lambda xy + \mu(z^2 + zx) = 0, \quad \Delta \equiv \lambda^2 \mu.$

If in (B) neither the three terms in x and z alone nor the three terms in y and z alone are factorable, we arrive at

(18) Set 10, $\lambda xy + \mu(x^2 + y^2 + z^2 + \alpha yz + \beta zx) = 0,$

$\alpha\beta \neq 0$, where $x^2 + z^2 + \beta zx$ and $y^2 + z^2 + \alpha yz$ are irreducible, with

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* We must have $g \neq 0$ or $f \neq 0$ or $fg \neq 0$, otherwise $C_2 = 0$ is a double line. If $g = 0$, $cf \neq 0$, we can put $x = y'$, $y = x'$, $z = z'$ and get the case $cg \neq 0$.

$$\Delta \equiv \mu[\lambda^2 + \alpha\beta\lambda\mu + (\alpha^2 + \beta^2)\mu^2].$$

If the quadratic factor of Δ is irreducible, then (18) may possibly be equivalent to (16). However in this case (4) must have $b_1 = c_1 = 0$, $a_1 \neq 0$, $a_2 = c_2 = 0$, $b_2 \neq 0$, $c_3 \neq 0$, $a_1^2 + a_3^2 + \beta a_1 a_3 = 0$ [because (16) has no term in x^2]. But this last equation has only the solution $a_1 = a_3 = 0$. If the quadratic factor of Δ is reducible, then (18) may possibly be equivalent to (15). However (4) must then have

$$a_1 a_2 = 0, \quad b_1 b_2 = 0, \quad a_1^2 + a_2^2 + a_3^2 + \alpha a_2 a_3 + \beta a_1 a_3 = 0,$$
$$b_1^2 + b_2^2 + b_3^2 + \alpha b_2 b_3 + \beta b_1 b_3 = 0,$$

since (15) has no terms in x^2 and y^2. But then we have $a_1 \neq 0$ (say), $a_2 = 0$, which gives us $a_1^2 + a_3^2 + \beta a_1 a_3 = 0$ (an irreducible equation).

If in (B) $c = 0$, $f \neq 0$, we have a pencil reducible to (17).

If in (B) $c = f = 0$, $b \neq 0$, we have

(19) Class 11, $\lambda xy + \mu(y^2 + zx) = 0, \quad \Delta \equiv \mu^3.$

If in (B) $c = f = b = 0$, we obtain

(20) Class 12, $\lambda xy + \mu xz = 0, \quad \Delta \equiv 0.$

If in (B) $f = 0$, $c \neq 0$, and the three terms in x and z alone in C_2 are not factorable, we get

(21) Class 13, $\lambda xy + \mu(x^2 + z^2 + \alpha zx) = 0, \quad \Delta \equiv \lambda^2\mu.$

The line pair $C_2 = 0$ in (21) is conjugate imaginary, so this pencil is non-equivalent to (17).

Set III. We put (3) in a form (C) with $C_1 = 0$ of the form (7) and with $h = 0$ in C_2. We must have $f \neq 0$ or $g \neq 0$ or $fg \neq 0$, otherwise $C_2 = 0$ is a double line. We suppose $g \neq 0$ (the case $g = 0$, $f \neq 0$ is reducible to this). We must also have $c \neq 0$, otherwise (C) contains a real line pair. We now transform (C) to a pencil (C') with $C_1' = 0$ of the form (7) and with $f' = h' = 0$, $c' \neq 0$, $g' = 1$. Also in (C') either $a' \neq 0$ or $b' \neq 0$ or $a'b' \neq 0$, otherwise $C_2' = 0$ is a real line pair.

If $b' \neq 0$ in (C'), we get

(23) Set 14, $\lambda(x^2 + y^2 + \beta xy) + \mu(y^2 + z^2 + \gamma xz) = 0,$

$\beta\gamma \neq 0$, $\Delta \equiv \mu(\beta^2\lambda^2 + \gamma^2\lambda\mu + \gamma^2\mu^2)$. We assume for this set that γ makes the quadratic factor of Δ irreducible.

If the quadratic factor of Δ of (23) is reducible, then we take for $C_1' = 0$ and $C_2' = 0$ in (C') two conjugate imaginary line pairs with vertices at $(0, 0, 1)$ and $(0, 1, 0)$ respectively.* We obtain

(24) Set 15, $\qquad \lambda(x^2 + y^2 + \alpha xy) + \mu(x^2 + z^2 + \beta xz) = 0$

$\alpha\beta \neq 0$, $C_1 = 0$ and $C_2 = 0$ reducible to (7), $\Delta \equiv \lambda\mu(\alpha^2\mu + \beta^2\mu)$. The third degenerate conic of (24), which must be a conjugate imaginary line pair, is

$$(\beta^2/\alpha^2 + 1)x^2 + (\beta^2/\alpha^2)y^2 + z^2 + (\beta^2/\alpha)xy + \beta xz = 0.\dagger$$

This reduces to the form (17) with $\alpha/(\alpha^2 + \beta^2)^{\frac{1}{2}}$ instead of α.

If $b' = 0$ in (C') we get Set 15 again.

We note that there is no pencil with just one (or with just two) conjugate imaginary line pairs for its degenerate conics.

Set IV. We reduce the discriminant of (3) to the form

(25) $$\Delta \equiv \lambda^3 + \lambda^2\mu + \alpha\mu^3,$$

which is to be an irreducible cubic. Then we put (3) in a form (D) with $C_1 \equiv x^2 + yz$. We must have $g \neq 0$ or $h \neq 0$ or $gh \neq 0$ in C_2 of (D), otherwise (D) has a double line. We suppose $g \neq 0$ (the case $g = 0$, $h \neq 0$ is reducible to this). Now we reduce (D) to a form (D') with $C_1' \equiv x^2 + b''y^2 + yz$ and with $f' = h' = 0$, $g' = 1$ in C_2', by transformations on x, y, z alone. The discriminant $\Delta \equiv \lambda^3 + b'\mu^3 + a'\lambda^2\mu + b''\lambda\mu^2$ must be of the form (25), so we have $b' = \alpha$, $a' = 1$, $b'' = 0$. We get

(26) Set 16, $\qquad \lambda(x^2 + yz) + \mu(x^2 + \alpha y^2 + \beta z^2 + xz) = 0,$

with (25) as its discriminant.

Suppose now that in $GF(2^n)$ we have $2^n = 3k + 1$ and the pencil (3) has a discriminant reducible to $\Delta \equiv \lambda^3 + \alpha\mu^3$, where $\alpha \neq$ cube. We put the discriminant in this form, we reduce $C_1 = 0$ to the form (5); then our pencil has a discriminant $\Delta \equiv \lambda^3 + \Delta_2\mu^3 + c\lambda^2\mu + (fg + h^2)\lambda\mu^2$, where Δ_2 belongs to $C_2 = 0$. So we must have $c = 0$, also $f \neq 0$ or $g \neq 0$ or $fg \neq 0$ (otherwise the pencil has a degenerate conic). We suppose $f \neq 0$ (the case $f = 0$,

* A pencil containing two conjugate imaginary line pairs with the same vertex contains also a double line.

† This set evidently does not exist in $GF(2)$.

$g \not\equiv 0$ is reducible to this). We perform some evident transformations and get the pencil into the form

$$\lambda(a_1x^2 + z^2 + xy) + \mu(a_2x^2 + b_2y^2 + yz) = 0,$$

with $\Delta \equiv \lambda^3 + a_2\mu^3 + a_1\lambda\mu^2$. So we have $a_1 = 0$, and we get

(27) Set 17, $\lambda(z^2 + xy) + \mu(\alpha x^2 + \beta y^2 + yz) = 0,$

$\Delta \equiv \lambda^3 + \alpha\mu^3$, $\alpha \not\equiv$ cube.

We now append a list of the classes of pencils of conics, together with their typical pencils and their discriminants.

Class 1, $\lambda x^2 + \mu y^2 = 0, \quad \Delta \equiv 0.$

Class 2, $\lambda x^2 + \mu(z^2 + xy) = 0, \quad \Delta \equiv \mu^3.$

Class 3, $\lambda x^2 + \mu(y^2 + xy) = 0, \quad \Delta \equiv 0.$

Class 4, $\lambda x^2 + \mu xy = 0, \quad \Delta \equiv 0.$

Class 5, $\lambda x^2 + \mu(y^2 + z^2 + \alpha yz) = 0, \quad \Delta \equiv \alpha^2\lambda\mu^2.$

Class 6, $\lambda x + \mu yz = 0, \quad \Delta \equiv \lambda\mu^2.$

Class 7, $\lambda xy + \mu(z^2 + yz + xz) = 0, \quad \Delta \equiv \lambda\mu(\lambda + \mu).$

Class 8, $\lambda xy + \mu(z^2 + y^2 + \alpha yz + xz) = 0, \quad \Delta \equiv \mu(\lambda^2 + \mu^2 + \alpha\lambda\mu).$

Class 9, $\lambda xy + \mu(z^2 + zx) = 0, \quad \Delta \equiv \lambda^2\mu.$

Set 10, $\lambda xy + \mu(x^2 + y^2 + z^2 + \alpha yz + \beta xz) = 0,$
$\Delta \equiv \mu[\lambda^2 + \alpha\beta\lambda\mu + (\alpha^2 + \beta^2)\mu^2].$

Class 11, $\lambda xy + \mu(y^2 + zx) = 0, \quad \Delta \equiv \mu^3.$

Class 12, $\lambda xy + \mu xz = 0, \quad \Delta \equiv 0.$

Class 13, $\lambda xy + \mu(x^2 + z^2 + \alpha xz) = 0, \quad \Delta \equiv \lambda^2\mu.$

Set 14, $\lambda(x^2 + y^2 + \beta xy) + \mu(y^2 + z^2 + \gamma xz) = 0,$
$\Delta \equiv \mu(\beta^2\lambda^2 + \gamma^2\lambda\mu + \gamma^2\mu^2).$

Set 15, $\lambda(x^2 + y^2 + \alpha xy) + \mu(x^2 + z^2 + \beta xz) = 0,$
$\Delta \equiv \lambda\mu(\alpha^2\lambda + \beta^2\mu).$

Set 16, $\lambda(x^2 + yz) + \mu(x^2 + \alpha y^2 + \beta z^2 + xz) = 0,$
$\Delta \equiv \lambda^3 + \lambda^2\mu + \alpha\mu^3.$

Set 17, $\lambda(z^2 + xy) + \mu(\alpha x^2 + \beta y^2 + yz) = 0,$
$\Delta \equiv \lambda^3 + \alpha\mu^3.$

SYRACUSE, NEW YORK.

On Generalized Lacunae.*

By John J. Gergen.

1. *Introduction.* The lacunae of a Taylor's series are those terms with zero coefficients; a series

$$T(x) = \sum_{m=0}^{\infty} c_m x^m$$

having lacunae can be written in form

$$T(x) = \sum_{n=0}^{\infty} c_{\lambda_n} x^{\lambda_n} \tag{1}$$

where $c_m = 0$ if $m \neq \lambda_n$, and $c_m \neq 0$ if $m = \lambda_n$. The effects of lacunae in series upon the nature of the singularities of the function defined by the series have been considered by many writers.** Thus, for example, Hadamard † has proved that if in series (1)

$$(\lambda_{n+1} - \lambda_n)/\lambda_n > \lambda > 0 \qquad (n = 0, 1, 2, \cdots),$$

then the circle $|x| = R$, where $R^{-1} = \overline{\lim} \, |c_{\lambda_n}|^{1/\lambda_n}$ and is supposed finite but different from zero, is a cut for the function $T(x)$. Fabry ‡ has generalized this theorem by proving that the same conclusion holds if

$$\lim \, (\lambda_{n+1} - \lambda_n) = \infty.$$

Mandelbrojt § has proved that if, for some integer $p \geq 0$

$$\overline{\lim} \, (\lambda_{n+1} - 2^p \lambda_n) = \infty,$$

then $T(x)$ cannot be written in the form

$$\phi(x)/[P(x)]^{1/(p+1)},$$

* Preliminary report presented to the American Mathematical Society, April 16, 1927. A résumé of the theorems appears in the *Comptes Rendus des Séances de l'Académie des Sciences*, t. 184 (1927), pp. 1040-1043.

** For a bibliography and discussion of the work that has been done on the series of Taylor and their analytic prolongation see Hadamard et Mandelbrojt, "La Série de Taylor et Son Prolongement Analytique," *Scientia*, No. 12, Paris (1926).

† Hadamard, *Journal de Mathématiques*, t. 8 (1892), p. 101.

‡ Fabry, *Acta Mathematica*, t. 22 (1898), p. 65.

§ Mandelbrojt, *Annales Scientifiques de l'École Normale Supérieure*, 3e série, t. 40 (1923), p. 413. Dr. Mandelbrojt suggested this problem on generalized lacunae.

where $\phi(x)$ is regular within a circle $|x| = R' > R$, and $P(x)$ is a polynomial. Tsuji * and Pólya † independently have generalized this theorem by showing that if

$$\overline{\lim}\ (\lambda_{n+1} - \lambda_n) = \infty,$$

then $T(x)$ has on the circle $|x| = R$ at least one singular point which is not an algebraic-logarithmic singular point.‡

It is the purpose here to determine facts about the singularities of functions defined by Taylor's series having geenralized lacunae. Mandelbrojt § defines the generalized lacunae of the series $\sum_0^\infty a_n x^n$, relative to the entire function $g(z)$, as the ordinary lacunae of the series $\sum_0^\infty g(a_n)x^n$. It should be noted that if one chooses $g(z)$ such that $g(0) = 0$, a series having ordinary lacunae can be made to have generalized lacunae with respect to this function $g(z)$. On the other hand, the fact that a series has generalized lacunae does not imply that it has ordinary lacunae.

2. *Preliminary Definition and Theorem.* Definition. Suppose β an essential singularity of $f(x)$ and that $f(x)$ is regular except at β. We say that β is a singularity of the exponential order q (≥ 0) if the entire ‖ function in x, $f((x\beta + 1)/x)$, is of the apparent order q.¶ If $f(x)$ is an entire function, the point at infinity is a singularity of the exponential order q if $f(x)$ is of the apparent order q.

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* Tsuji, *Japanese Journal of Mathematics*, Vol. 3 (1926), p. 69.

† Pólya, *Comptes Rendus des Séances de l'Académie des Sciences*, t. 184 (1927), p. 502.

‡ The function $f(x)$ is said to have at the point x_0 an algebraic-logarithmic singular point if it can be represented in a circle C with center x_0 by the sum of a finite number of terms of the form

$$(x - x_0)^s\ [log\ (x - x_0)]^q \phi(x),$$

where s is a rational number, q a non-negative integer, and $\phi(x)$ is regular in C.

§ Mandelbrojt, *Bulletin de la Société Mathématique de France*, t. 53 (1925), p. 235. The above definition is less general than that given by Mandelbrojt but is sufficient for our purposes.

‖ We do not classify polynomials as entire functions. See Borel, *Leçons sur les Fonctions Entières*, 2nd ed., Paris (1921), p. 1.

¶ The entire function $f(z)$ is of the apparent order q if corresponding to an arbitrary $\epsilon > 0$ there exists an r_ϵ such that if $|z| = r > r_\epsilon$ then

$$\text{maximum of } |f(z)| \text{ on } |z| = r < e^{r^{q+\epsilon}}$$

and if q is the smallest number having this property. See Borel, *loc. cit.*, p. 75.

Since for x^t, where t is an arbitrary positive integer, we have

$$| x^t | < e^{|x|^\epsilon}$$

for $| x |$ sufficiently large, ϵ being arbitrarily small, we shall say that β is a singular point of $f(x)$ of the exponential order zero if it is a pole of $f(x)$ and $f(x)$ is regular except at β.

Finally, if $F(x) = f(x) + \phi(x)$ and if $\phi(x)$ is regular at β, we shall say that β is a singular point of $F(x)$ of the exponential order q if it is that for $f(x)$.

We then have the preliminary theorem.

THEOREM I. *Let* $f(x) = \sum_0^\infty a_n x^n$ *be a uniform function having as its only possible singularities the pth roots of unity and the point at infinity. Suppose that no singular point of* $f(x)$ *on the circle* $| x | = 1$ *is of the exponential order greater than q.* Let* $g(z) = \sum_1^\infty \alpha_m z^m$ *be an entire function such that* †

$$| \alpha_m |^{1/m} = 0 \ (e^{-2^{(q+1)m}})$$

Then the function $G(x) = \sum_0^\infty g(a_n) x^n$ *has as its only possible singularities the pth roots of unity and the point at infinity.*‡

For the function $g(z)$ we might take

$$\sum_1^\infty z^m / m ! e^{m 2^{(q+1)m}}$$

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* This assumption implies that if β_k is singular point of $f(x)$, we can write

$$f(x) = \xi_k(x) + \Gamma(x)$$

where $\xi_k(x)$ is regular except at the point β_k and $\Gamma(x)$ is regular at β_k. This is possible if $f(x)$ is uniform.

† This is the notation of Landau. In this instance it means that

$$\lim_{m=\infty} | a_m |^{1/m} \ e^{2^{(q+1)m}} = 0.$$

‡ For theorems of similar nature see Hadamard et Mandelbrojt, *loc. cit.*, pp. 48-53. See also Soula, *Journal de Mathématiques*, t. 86 (1921), p. 94, and Leau, *Comptes Rendus des Séances de la Société Mathématique de France*, t. 25-26 (1897-98), p. 267, and *Journal de Mathématiques*, t. 5 (1899), p. 365. For the case where one has

$$\lim (x - \beta)^q f(x) = 0$$

for all the singularities β, Leau replaces the above condition on $g(z)$ by one much less restrictive. The above theorem is proved in a manner similar to Leau's.

Proof: Consider the sequence of functions $\{f_m(x)\}$, where

$$f_1(x) = f(x) = \sum_0^\infty a_n x^n,$$

$$f_m(x) = H(f_1, f_{m-1}) = \sum_{n=0}^\infty a_n{}^m x^n \qquad (m = 2, 3, \cdots).$$

By Hadamard's * theorem on the multiplication of singularities $f_m(x)$ has as its only possible singularities the set β consisting of

$$\beta_k = e^{2\pi i(k/p)} \qquad (k = 0, 1, \cdots, p-1),$$

and γ the point at infinity. Moreover, $f_m(x)$ is a uniform function.†

Let $\epsilon_1 > 0$ be arbitrarily small but $< |\beta_1 - \beta_0|/2$ if $p > 2$, or $< 1/2$ if $p \leq 2$. Let R_1 be arbitrarily large but $> 1 + \epsilon_1$. Let D_1 be the open region lying exterior to the circles $|x - \beta_k| = \epsilon_1$ but interior to the circle $|x| = R_1$. We desire to prove that the series

$$(2) \qquad \alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_m f_m(x) + \cdots$$

converges uniformly in D_1.

Consider the uniform functions $\phi(x) = \sum_0^\infty u_n x^n$, $\psi(x) = \sum_0^\infty v_n x^n$ having as their only possible singularities the set β. Let $\eta > 0$ be arbitrarily small but $\leq \epsilon_1$ and σ be arbitrarily large but $\geq R_1$. Let $\eta' = (1+\eta)^{\frac{1}{2}} - 1$, $\sigma' = \sigma/(1-\eta)$. Construct a circle K', $|x| = \sigma'$. Construct the circles b_k', $|x - \beta_k| = \eta'$. Let S' be the open region lying exterior to the circles b_k' but interior to the circle K' and let T' be its boundary. Through the points of intersection of b_k' with the circle $|x| = 1$ construct radii to K'. Let l_k', l_k'' be the segments of these radii from $|x| = 1$ to $|x| = \sigma'$, where

$$\text{angle of inclination of } l_k' = \text{amplitude } \beta_k + \omega,$$
$$\text{angle of inclination of } l_k'' = \text{amplitude } \beta_k - \omega.$$

If we regard the segments l_k' as forming cuts of two sides the curve K_1 consisting of T' and the segments l_k' bounds a simply connected open region S_1. If x is any point in S_1 and z any point on K_1, then x/z is not a point of β, for if it were, we should have either $x = z\beta_k$, whence x would be a point of K_1, or else x would be the point at infinity. The function

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* Hadamard, *Acta Mathematica*, t. 22 (1898), p. 55.

† Borel, *Comptes Rendus des Séances de la Société Mathématique de France*, t. 25-26 (1897-98), p. 238.

$$F_1(x) = \frac{1}{2\pi i} \int_{K_1} \phi(z)\, \psi\left(\frac{x}{z}\right) \frac{dz}{z}$$

is then as in Hadamard's theorem the analytic prolongation of

$$\sum_0^\infty u_n v_n x^n \tag{3}$$

for x in S_1. Moreover, for these values of x we have

$$F_1(x) = \frac{1}{2\pi i} \int_{T'} \phi(z)\, \psi\left(\frac{x}{z}\right) \frac{dz}{z} ,$$

since the integral over the segment l_k' is cancelled when the direction of integration is reversed.

Similarly, the curve K_2 consisting of T' and the segments l_k'' bounds a simply connected region S_2, and the function

$$\begin{aligned} F_2(x) &= \frac{1}{2\pi i} \int_{K_2} \phi(z)\, \psi\left(\frac{x}{z}\right) \frac{dz}{z} \\ &= \frac{1}{2\pi i} \int_{T'} \phi(z)\, \psi\left(\frac{x}{z}\right) \frac{dz}{z} \end{aligned}$$

is the analytic prolongation of (3) for x in S_2. But $F_1(x) = F_2(x)$ for x in the common parts of S_1 and S_2, since $\phi(x)$ and $\psi(x)$ are uniform functions. Hence $F_1(x)$ can be prolonged analytically across the segments l_k' and its value at each point of S' is that of the integral

$$F_1(x) = \frac{1}{2\pi i} \int_{T'} \phi(z) \psi\left(\frac{x}{z}\right) \frac{z}{dz} \quad .$$

Hence $H(\phi, \psi)$, the analytic prolongation of (3), is given for x in S' by the formula

$$H(\phi, \psi) = \frac{1}{2\pi i} \int_{T'} \phi(z)\, \psi\left(\frac{x}{z}\right) \frac{dz}{z} \quad . \tag{4}$$

The relation (4) is true in particular in the region S lying interior to the circle $|x| = \sigma$ but exterior to the circles $|x - \beta_k| = \eta$. Moreover, for x in S and z on T' we have x/z in S'. For if z is on K', then for all x in S

$$|x/z| < \sigma/\sigma' = 1 - \eta < 1 - \eta',$$

while if z is on some b_k',

$$|x/z| < \sigma/(1 - \eta') < \sigma/(1 - \eta) = \sigma'$$

and
$$| (x/z) - \beta_j | = | (x - z\beta_j)/z | > (\eta - \eta')/(1 + \eta') = \eta',$$

since $z\beta_k$ is a point on one of the circles b_k'.

If then

$$| \phi(z) | < N' \text{ for } z \text{ on } T'$$

and
$$| \psi(x) | < N'' \text{ for } x \text{ in } S',$$

it follows that

$$| H(\phi, \psi) | < \frac{\sigma' + \eta' p}{1 - \eta'} N'N'' \tag{5}$$

for all x in S.

The inequality (5) permits us to demonstrate that the series (2) converges uniformly in D_1. Let

$$\epsilon_r = (1 + \epsilon_{r-1})^{\frac{1}{2}} - 1 = (1 + \epsilon_1)^{1/2^{r-1}} - 1 \qquad (r = 2, 3, \cdots),$$
$$R_r = R_{r-1}/(1 - \epsilon_{r-1}) \qquad (r = 2, 3, \cdots).$$

Let D_r be the open region interior to the circle $| x | = R_r$ but exterior to the circles $| x - \beta_k | = \epsilon_r$. Let C_r be the boundary of D_r.

Since $f_1(x)$ is holomorphic in D_r and continuous on its boundary there exists an M_r independent of x in D_r and on C_r such that for these values of x

$$| f_1(x) | < M_r \qquad (r = 1, 2, 3, \cdots).$$

It follows that in the region D_1 the series

$$\sum_2^\infty \alpha_m f_m(x) \ll \sum_2^\infty | \alpha_m | \; L_2 \cdot L_3 \cdots L_m \, M_m^m, \tag{6}$$

where
$$L_r = (R_r + \epsilon_r p)/(1 - \epsilon_r)$$

for we can conclude that

$$| f_m(x) | < L_2 \cdot L_3 \cdots L_m \, M_m^m$$

for these values of x by applying the results obtained for $\phi(x)$ and $\psi(x)$, replacing $\psi(x)$ in turn by $f_1(x)$, $f_2(x)$, $\cdots$, $f_{m-1}(x)$; $\phi(x)$ in each case by $f_1(x)$; the region S' in turn by D_m, D_{m-1}, $\cdots$, D_2; the region S in turn by D_{m-1}, D_{m-2}, $\cdots$, D_1; N' in each case by M_m; etc.

We note that

$$\epsilon_1 = (1 + \epsilon_1) - 1 = \epsilon_r [(1 + \epsilon_1)^{(2^{r-1}-1)/2^{r-1}} + \cdots + 1],$$

so that

$$\epsilon_1/2^r < \epsilon_r < 1/2^r \qquad (r = 1, 2, \cdots).$$

Moreover,

$$\begin{aligned} R_r &= R_1 \prod_1^{r-1} [1/(1-\epsilon_s)] \\ &< R_1 \prod_1^{\infty} [1/(1-\epsilon_s)] \\ &< R_1 \prod_1^{\infty} [1 + 1/(2^s - 1)] \\ &< R, \end{aligned}$$

where R is finite and independent of r.

Hence

$$L_r < (R+p)/(1-\epsilon_r)$$

and

$$\begin{aligned} L_2 \cdot L_3 \cdot \cdots \cdot L_m &< (R+p)^m \prod_1^{\infty} [1/(1-\epsilon_s)] \\ &< [R(R+p)]^m. \end{aligned}$$

We can then replace (6) by

$$(7) \qquad \sum_2^{\infty} \alpha_m f_m(x) \ll \sum_2^{\infty} |\alpha_m| [R(R+p)M_m]^m.$$

In order to find a bound for M_m we write

$$f_1(x) = \xi_0(x) + \xi_1(x) + \cdots + \xi_{p-1}(x) + \xi_\gamma(x),$$

where $\xi_k(x)$ is regular except possibly at the point β_k and $\xi_\gamma(x)$ is regular except possibly at the point at infinity. The values of $|\xi_k(x)|$ for which $|x - \beta_k| \geq \rho$ are the values of $|\xi_k[(x\beta_k + 1)/x]|$ for which $|x| \leq 1/\rho$. Then, since every singular point of $f(x)$ is of the exponential order $\leq q$, we have for $|x| \leq 1/\rho$,

$$|\xi_k (x\beta_k + 1)/x| < N_k e^{(1/\rho)^{q+\epsilon_1}} \qquad (k = 0, 1, \cdots, p-1),$$

where N_k is independent of x. Consequently,

$$|\xi_k(x)| < N_k e^{(1/\rho)^{q+\epsilon_1}}$$

for $|x - \beta_k| \geq \rho$. Hence, for $|x - \beta_k| \geq \epsilon_m$

$$\begin{aligned} |\xi_k(x)| &< N_k e^{(1/\epsilon_m)^{q+\epsilon_1}} \\ &< N_k e^{(2^m/\epsilon_1)^{q+\epsilon_1}}. \end{aligned}$$

Let N_γ be a bound for $|\xi_\gamma(x)|$ for $|x| \leq R$. Let

$$\bar{N} = N_1 + N_2 + \cdots + N_{p-1} + N_\gamma,$$

then

$$M_m < \bar{N}\, e^{(2^m/\epsilon_1)^{q+\epsilon_1}}$$
$$< \bar{M}\, e^{2^{(q+1)m}},$$

where $\bar{M}$ is independent of m.

We may then replace (7) by

$$\sum_2^\infty \alpha_m f_m(x) \ll \sum_2^\infty |\, \alpha_m \,| M^m\, e^{m2^{(q+1)m}}, \tag{8}$$

where $M = \bar{M}\,(R+p)R$.

By hypothesis

$$|\, \alpha_m \,|^{1/m} = 0\ (e^{-2^{(q+1)m}}).$$

Hence the series

$$\sum_2^\infty \alpha_m\, e^{m2^{(q+1)m}} x^m$$

converges absolutely for all values of x. It follows that $\sum_1^\infty \alpha_m f_m(x)$ converges uniformly to a function $G(x)$, holomorphic in D_1. Moreover, by a well-known theorem of Weierstrass

$$\sum_1^\infty \alpha_m f_{(x)}^{(n)} \qquad (n = 2, 3, \cdots),$$

converges uniformly to $G^{(n)}(x)$ in D_1. Hence

$$G(0) = \sum_{m=1}^\infty \alpha_m f_m(0) = \sum_{m=1}^\infty \alpha_m a_0{}^m = g(a_0)$$

and in general

$$G^{(n)}(0) = \sum_{m=1}^\infty \alpha_m f_m{}^{(n)}(0) = n!.\sum_{m=1}^\infty \alpha_m a_n{}^m = n!\, g(a_n).$$

We have then as the unique expansion of $G(x)$ at the origin

$$G(x) = \sum_0^\infty G^{(n)}(0)\, x^n/n!$$

$$= \sum_0^\infty g(a_n) x^n.$$

The function $G(x)$ was proved to be holomorphic in D_1. But since ϵ_1 was arbitrarily small and R_1 arbitrarily large its only possible singularities are the pth roots of unity and the point at infinity.

3. *Notes on the Preliminary Theorem.*

a. An entire function $g(z)$ satisfying the condition of Theorem I is of the apparent order zero since *

$$\lim_{m=\infty} \frac{\log\ (m!)}{\log\ |\ \alpha_m\ |} = 0.$$

It is then of the class zero † and can be written in the form

$$g(z) = A\, z^t \prod_1^\infty [1 - (z/\mu_n)],$$

where A is a constant independent of z, t a positive integer and μ_n the nth zero of $g(z)$, it being supposed that $0 < |\ \mu_n\ | \leq \mu_{n+1}$.

If we suppose $g(z)$ a polynomial of degree m,

$$g(z) = \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_m z^m,$$

then

$$g(a_n) = \alpha_1 a_n + \alpha_2 a_n{}^2 + \cdots + \alpha_m a_n{}^m$$

and

$$\sum_2^\infty g(a_n)x^n = \alpha_1 \sum_2^\infty a_n x^n + \alpha_2 \sum_0^\infty a_n{}^2 x^n + \cdots + \alpha_m \sum_0^\infty a_n{}^m x^n.$$

The Theorem I is then true in this case even though we suppose nothing about the order of increasing of $f(x)$ in the neighborhood of its singular points. However, in the theorems that follow we shall consider $g(z)$ an entire function rather than a polynomial.

b. If in Theorem I the only possible singular points of $f(x)$ are the pth roots of unity, the point at infinity being a regular point, then $G(x)$ is regular at the point at infinity. This follows immediately from a theorem of Weierstrass since $f_m(x)$ is continuous on the circle $|\ x\ | = \rho > 1$, is regular for $|\ x\ | > \rho$, and the series (2) converges uniformly on $|\ x\ | = \rho$.

If $f(x)$ is an entire function, then $f_m(x)$ is likewise an entire function and we can replace M_r by a number Q independent of $|\ x\ | \leq R$ and of r. We then have instead of (8)

$$\sum_2^\infty \alpha_m f_m(x) \ll \sum_2^\infty |\ \alpha_m\ |\ (QM)^m$$

for $|\ x\ | < R_1$, which is sufficient to prove that if $g(z)$ is simply an entire function, then $G(x)$ is regular except possibly at the point at infinity.

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* Hadamard et Mandelbrojt, *loc. cit.*, p. 50.

† Borel, *Fonctions Entières, loc. cit.*, p. 76.

c. If in Theorem I the only possible singularities of $f(x)$ are the pth roots of unity then either $G(x)$ reduces to the constant $g(a_0)$ or else it has a singularity at one of the pth roots of unity. It is sufficient for the latter to be true that for a single a_n, $n \neq 0$, $g(a_n) \neq 0$.

4. *Theorems on Generalized Lacunae.* We shall designate by $\{\lambda_n\}$, $\{\lambda'_n\}$ two increasing, infinite sequences of positive integers complimentary with regard to the sequence $\{n\}$. If there exists a series

$$\phi(x) = \sum_{n=0}^{\infty} b_{\lambda'_n} x^{\lambda'_n},$$

with radius of convergence equal to one and having on the circle $|x| = 1$ an algebraic-logarithmic singular point or an isolated singlar point around which $\phi(x)$ is uniform, we shall designate $\{\lambda'_n\}$ by $S_{\lambda'_n}$.

The sequence $\{\lambda'_n\} = \{nk\}$, where k is a positive integer > 1, would be properly designated by $S_{\lambda'_n}$, for the series

$$1/(1-x^k) = \sum_{0}^{\infty} x^{nk}$$

has a pole on the circle $|x| = 1$.

The coefficients of a series

$$\psi(x) = \sum_{0}^{\infty} c_n x^n$$

having the subscripts of a sequence $S_{\lambda'_n}$ we shall designate by $S(c_{\lambda'_n})$.

A theorem proved by Mandelbrojt * and generalized by Pólya † states that if in the series

$$\psi(x) = \sum_{0}^{\infty} c_n x^n$$

with radius of convergence equal to one, each term of a sequence $S(c_{\lambda'_n})$ is zero, then $\psi(x)$ has at least two singular points on the circle $|x| = 1$.

We shall designate by Eg, q the set of zeros of an entire function

$$g(z) = \sum_{m=1}^{\infty} \alpha_m z^m$$

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* Hadamard et Mandelbrojt, *loc. cit.*, p. 99.

† Pólya, *loc. cit.*

having the property that for a certain number q

$$| \alpha_m |^{1/m} = 0 \ (e^{-2(q+1)m}).$$

We then have

THEOREM II. *Suppose the function* $f(x) = \sum_1^\infty a_n x^n$ *is regular for* $|x| < 1$ *and has a singular point at* $x = 1$. *Suppose that the terms of a sequence* $S(a_{\lambda'_n})$ *take only the values contained in an Eg, q and that for some* $\lambda_n \neq 0$, $g(a_{\lambda_n}) \neq 0$, *then either* $f(x)$ *has at least two singular points in the extended plane or else the point* 1 *is a singularity of the exponential order greater than* q.

In a word, we suppose that the terms for which $n = \lambda'_n$ are generalized lacunae of $\sum_2^\infty a_n x^n$, relative to the entire function $g(z)$. The proof is immediate. If $f(x)$ is regular except at the point 1, it is a uniform function, and if this point is a singularity of the exponential order $\leq q$, then the function $G(x) = \sum_0^\infty g(a_n) x^n$ is regular except at the point 1, which must be a singular point.

On the other hand, each term of $S(g(a_{\lambda'_n})$ is zero. Hence by the theorem of Mandelbrojt and Pólya stated above the function $G(x)$ has at least two singular points on the circle $|x| = 1$. It follows that either the point 1 is a singular point of the exponential order greater than q or else $f(x)$ has at least two singular points in the extended plane.

Finally, we have

THEOREM III. *Given the function* $f(x) = \sum_0^\infty a_n x^n$, *with integral coefficients* * *and radius of convergence equal to one. Suppose that the terms of a sequence* $S(a_{\lambda'_n})$ *take only the values contained in a set Eg, q. Suppose that* $\overline{\lim} \ | g(a_{\lambda_n}) |^{1/\lambda_n} = 1$ *and that* $\lim (\lambda_{n+1} - \lambda_n) = \infty$. *Then the circle* $|x| = 1$ *is a cut for the function* $f(x)$.

Here again we are supposing that the terms of $\sum_0^\infty a_n x^n$ for which $n = \lambda'_n$ are generalized lacunae of this series, relative to $g(z)$.

We have first, because of the hypotheses as to integral coefficients and unit radius of convergence, by a theorem of Carlson † and Pólya ‡ that either the circle $|x| = 1$ is a cut or else we can write $f(x)$ in the form

* a_n is said to be integral if $a_n = b_n + c_n i$, where b_n and c_n are integers.

† Carlson, *Mathematische Zeitschrift*, Bd. 9 (1921), p. 1.

‡ Pólya, *Mathematische Annalen*, Bd. 77 (1916), p. 497.

$$f(x) = \frac{P(x)}{(1-x^p)^t} \quad ,$$

where $P(x)$ is a polynomial and p, t are positive integers. Hence, if we suppose that the circle $|x| = 1$ is not a cut, the only possible singularities of $G(x) = \sum_0^\infty g(a_n)x^n$ are the pth roots of unity and the point at infinity. Moreover, $G(x)$ has at least one singular point on the circle $|x| = 1$.

On the other hand, by the theorem of Hadamard and Fabry stated in the Introduction the circle $|x| = 1$ is a cut for the function $G(x)$. It follows that

$$f(x) \neq \frac{P(x)}{(1-x^p)^t} \quad .$$

Hence the statement of the theorem is justified.

THE RICE INSTITUTE.

Properties of Certain Aggregate Functions.*

By Lester S. Hill.

1. *Introductory Notes.* Many mathematical papers have made use of arguments reposing upon the idea that rather general aggregates may conveniently be regarded as functions of other general aggregates. In fact, the notion is merely that of establishing a correspondence between two sets of aggregates, and is thus by no means new; the ordinary functions of elementary analysis, when considered geometrically, involve precisely this notion.

R. L. Moore, in a recent study of the structure of a continuum, was led to propose for consideration what he called "upper semi-continuous collections of continua," and to develop interesting properties of such collections.† But, while the term "upper semi-continuous" is very suggestive of ordinary functional relations, Dr. Moore does not introduce a variable. H. Hahn has, in like manner, closely approximated the formal consideration of upper-semi-continuous functions in which the function values are aggregates of points.‡

W. A. Wilson, in a recent paper,§ has formally introduced a variable with a definite range, and used a specific upper semi-continuous function in the study of continua irreducible between two points.

In the present paper, an attempt is made to develop certain fundamental properties of the aggregate functions thus appearing, implicitly and explicitly, in current investigations of analysis situs. Correspondences in which aggregates of points are functions of the points of an aggregate are considered from the standpoint of the theory of functions of real variables, and a few essential theorems are derived. These theorems constitute generalizations of standard relations in analysis. Thus, for the functions of the general class treated, we may assert that every monotone descending and convergent sequence of

* Communicated by title at the Columbus, Ohio, meeting of the American Mathematical Society, September, 1926.

† R. L. Moore, "Concerning Upper Semi-Continuous Collections of Continua," *Transactions of the American Mathematical Society*, Vol. 27 (1925), pp. 416-428; also, *Proceedings of the National Academy of Sciences*, Vol. 10 (1924), No. 8.

‡ H. Hahn, "Ueber irreduzible Kontinua," *Sitzungsberichte der Akademie der Wissenschaften*, Wien, Vol. 130 (1921).

§ W. A. Wilson, "On the Structure of a Continuum, Limited and Irreducible between Two Points," *American Journal of Mathematics*, Vol. 48 (1926), No. 3.

upper semi-continuous functions defines an upper semi-continuous function; that every sequence of continuous functions which is uniformly convergent, or which is merely "semi-uniformly" convergent, defines a continuous function; that every function which is upper semi-continuous, or lower semi-continuous, at each point of the set T is at most pointwise discontinuous in T when T is either a bounded region or a bounded perfect set.

But, aside from their intrinsic interest, the theorems presented in the following pages have the merit of providing a convenient basis for a study of very general types of regional frontier. In a paper soon to be offered for publication, it will be shown that certain upper semi-continuous transformations of the circumference of a circle into the frontier F of a bounded plane region can be extended, whenever F is the sum of two rather general continua irreducible between two points, so as to yield homeomorphic transformations of the bounded region into the interior of the circle.

2. *Terms and Notations. Limit Sets.* Let x be any point of the Euclidean space E; let T be a set of points in E; let τ be a point of T. These relations are expressed by $\tau \subseteq T \subseteq E$, or by $E \supseteq T \supseteq \tau$. If ϵ is any positive number ($\epsilon > 0$), we denote by $U_\epsilon(x)$ the set of all points of E whose distances from x are less than ϵ; by $V_\epsilon(\tau)$ the divisor of $U_\epsilon(\tau)$ and T; and by $V_\epsilon^*(\tau)$ the set obtained by omitting the point τ from $V_\epsilon(\tau)$.

The divisor of two sets A and B will be indicated as the logical product $A \cdot B$, and the union as the logical sum $A + B$. By $\bar{A}$ we shall denote the sum of A and its derived set.

If with each point t of a set T lying in the Euclidean space E there be associated in any definite way a set X_t of points lying in the Euclidean space G, we shall say that there is defined the "function" $X_t = f(t)$ of the "variable" t over the "range" T.

Let τ be a point of the range T. Let x be any point of G such that, for every $\epsilon > 0$, and every $\delta > 0$, $U_\epsilon(x)$ contains a point of $X_t = f(t)$ for some t in $V_\delta(\tau)$. If K is the class of points in G satisfying this condition, we say that K is the *upper* * *limit* of $X_t = f(t)$ as $t \to \tau$, and we write

$$K = \overline{\operatorname{Lim}_{t \to \tau}} X_t = \overline{\operatorname{Lim}_{t \to \tau}} f(t).$$

Let x be any point of G such that, for every $\epsilon > 0$, there exists a $\delta > 0$, for which $U_\epsilon(x)$ contains a point of $X_t = f(t)$ if t is any point of $V_\delta(\tau)$. If L is the class of points x in G satisfying this condition, we say that L is

* The upper and lower limit sets here defined are natural extensions of the "en-

the *lower limit* as $t \to \tau$ of $X_t = f(t)$, and we write $L = \underset{t\to\tau}{\underline{\text{Lim}}}\, X_t = \underset{t\to\tau}{\underline{\text{Lim}}}\, f(t)$.

If $K = L$, we say that $X_t = f(t)$ "converges" to the *limit* $K = L = \underset{t\to\tau}{\text{Lim}}\, X_t = \underset{t\to\tau}{\text{Lim}}\, f(t)$.

If $[t_i]$, $(i = 1, 2, 3, \cdots)$, is a denumerable sequence of points of T, the notations $\overline{\underset{i\to\infty}{\text{Lim}}}\, X_{t_i}$, $\overline{\underset{i\to\infty}{\text{Lim}}}\, X_{t_i}$, $\underset{i\to\infty}{\text{Lim}}\, X_{t_i}$ are almost self-explanatory.*

It is easy to see that the set $\underset{t\to\tau}{\underline{\text{Lim}}}\, X_t$, or the set $\underset{i\to\infty}{\underline{\text{Lim}}}\, X_{t_i}$, may be void (may contain no point). But we note:

THEOREM 1. *If the union of all the X_t sets in G is a bounded set, $\overline{\underset{t\to\tau}{\text{Lim}}}\, X_t$ will contain at least one point: $\overline{\underset{t\to\tau}{\text{Lim}}}\, X_t \neq 0$.*

It is to be observed that some or all of the X_t sets may consist of single points, and that some or all of these sets may be identical.

3. *Continuity and Semi-Continuity of the Function $X_t = f(t)$.* Let τ be a proper limit point, in the ordinary sense, of T. We shall call the function $X_t = f(t)$ upper semi-continuous, supracontinuous, lower semi-continuous, infracontinuous, respectively, at $t = \tau$, according as

$$X_\tau \supseteq \overline{\underset{t\to\tau}{\text{Lim}}}\, X_t,\ X_\tau = \overline{\underset{t\to\tau}{\text{Lim}}}\, X_t,\ X_\tau \subseteq \underset{t\to\tau}{\underline{\text{Lim}}}\, X_t,\ X_\tau = \underset{t\to\tau}{\underline{\text{Lim}}}\, X_t$$

the notations $A \subseteq B$, $B \supseteq A$ implying that every point of A is a point of B. We may agree that $X_t = f(t)$ possesses each of the four types of semi-con-

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semble d'accumulation" and the "ensemble limite" of Z. Janiszewski. Cf. "Sur les continus irreductibles entre deux points," *Journal de l'École Polytechnique*, Ser. II, Vol. 16 (1912).

* Let $[X^{(n)}]$ be a sequence of sets lying in the Euclidean space G, and depending upon one index n which ranges over a denumerable sequence of values $[n]$. Then $\overline{\underset{n\to\infty}{\text{Lim}}}\, X^{(n)}$ is the class of all points x of G having the property that $U_\epsilon(x)$, for any $\epsilon > 0$, contains points of $X^{(n)}$ for an infinity of values of n; and $\underset{n\to\infty}{\underline{\text{Lim}}}\, X^{(n)}$ is the class of all points x of G having the property that $U_\epsilon(x)$, for any $\epsilon > 0$, contains points of $X^{(n)}$ for all except (at most) a finite set of values of n. The set $\underset{n\to\infty}{\text{Lim}}\, X^{(n)}$ exists, and is the same as $\overline{\underset{n\to\infty}{\text{Lim}}}\, X^{(n)}$, when the latter set is the same as $\underset{n\to\infty}{\underline{\text{Lim}}}\, X^{(n)}$.

tinuity at an isolated point of T. Evidently, supra-continuity implies upper semi-continuity, and infracontinuity implies lower semi-continuity.

$X_t = f(t)$ is continuous at $t = \tau$ if and only if it possesses there all four types of semi-continuity. It is therefore continuous at an isolated point of T; and it is continuous at the proper limit point τ of T if and only if

$$X_\tau = \operatorname*{Lim}_{t\to\tau} X_t = \overline{\operatorname*{Lim}_{t\to\tau}} X_t = \underline{\operatorname*{Lim}_{t\to\tau}} X_t.$$

A simple example, in two-way space, will be sufficient.

Example: In the ordinary Cartesian plane, with rectangular coordinates, let T be the set of points $-\pi/2 \leqq x \leqq 3$, $y = 0$; and for each point of T, write $t = x$.

We define a function $X_t = f(t)$ over the range T as follows:

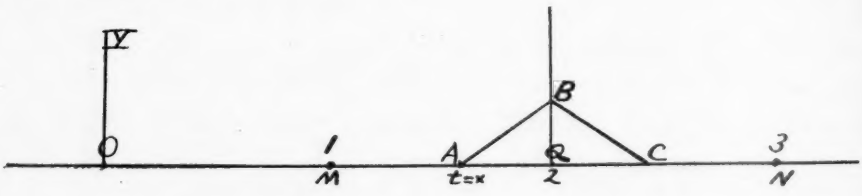
(1) At $t = -\pi/2$, X_t consists of the two points $x = -\pi/2$, $y = 1, -1$.

(2) For $-\pi/2 < t < 0$, X_t is the point $x = t$, $y = \sin t$.

(3) At $t = 0$, X_t is the segment of the Y-axis: $x = 0$, $-1 \leqq y \leqq 1$.

(4) For $0 < t \leqq 1$, X_t consists of the two points: $x = t$, $y = \sin(1/t)$, $y = \cos(1/t)$.

(5) For $1 < t < 2$, X_t consists of the closed set of all points of the broken line $A\,B\,C$.



with $AB = BC$, and $\tan BAC = (2-t)/(t-1) = AQ/MA$.

(6) At $t = 2$, X_t is the point $x = 2$, $y = 0$.

(7) For $2 < t < 3$, $X_t = X_{t_1}$ with $t_1 = 4 - t$.

(8) At $t = 3$, X_t consists of all points: $x = 3$, $y \geqq 0$.

This function is illustrated in the figure for the interval ($t = x = 1$ to $t = x = 3$) only. If t' and t'' are two points of that interval which are symmetrically located with respect to $t = 2$, then $X_{t'} = X_{t''}$.

There are only four points in T at which the function is not continuous.

At $t = 0$, it is supracontinuous; at $t = -\pi/2$, it is merely upper semi-continuous. At $t = 1$, it is infracontinuous; at $t = 3$, it is merely lower semi-continuous.*

4. *Concerning the Limit Sets of $X_t = f(t)$.* Several theorems expressing useful or significant properties of the limit sets of $X_t = f(t)$ will be recorded without proof.

THEOREM 2. *Let τ be a limit point of the range $T = \{t\}$. For some $\delta > 0$, let the union of the X_t sets, as t ranges over $V_\delta^*(\tau)$, be bounded. Let $K = \overline{\operatorname{Lim}}_{t\to\tau} X_t$, $L = \underline{\operatorname{Lim}}_{t\to\tau} X_t$. Then, for every $\epsilon > 0$, there exists a $\delta_\epsilon > 0$, such that X_t lies in $U_\epsilon(K)$, and L lies in $U_\epsilon(X_t)$ whenever t lies in $V_{\delta_\epsilon}^*(\tau)$.*

Since $X_t = f(t)$ converges to a limit set, as $t \to \tau$, if and only if $K = L$, Theorem 2 serves to emphasize the essentially bipartite character of convergence. The following theorem throws some further light upon the nature of convergence.

THEOREM 3. *Under the hypotheses of Theorem 2, $X_t = f(t)$ will converge as $t \to \tau$, if and only if there exists a set M having the property that for every denumerable sequence $[t_i]$ of points in T converging to τ, $M = \overline{\operatorname{Lim}}_{i=\infty} X_{t_i}$; and if the set M exists, $M = \operatorname{Lim}_{t\to\tau} X_t$.*

As might be expected, $\overline{\operatorname{Lim}}_{t\to\tau} X_t$ and $\underline{\operatorname{Lim}}_{t\to\tau} X_t$ can be related, respectively, to a divisor set and a union set.

THEOREM 4. *Under the hypotheses of Theorem 2, let B be any closed set in the space G having the property that, for any $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that, when t lies in $V_{\delta_\epsilon}^*(\tau)$, the set X_t lies in $U_\epsilon(B)$; and let C be any set in G having the property that, for any $\epsilon > 0$, there exists a*

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* It is to be noted that: $\overline{\operatorname{Lim}}_{t\to1} X_t = \overline{\operatorname{Lim}}_{t\to3} X_t = \underline{\operatorname{Lim}}_{t\to3} X_t$ consists of all the points $x = 1$, $y \geqq 0$ and all the points $x = 3$, $y \geqq 0$; while $\underline{\operatorname{Lim}}_{t\to1} X_t$ consists of the two points $x = 1$, $y = \sin 1$ and $x = 1$, $y = \cos 1$. Notwithstanding that $\overline{\operatorname{Lim}}_{t\to3} X_t = \underline{\operatorname{Lim}}_{t\to3} X_t$, the function $X_t = f(t)$ is not continuous at $t = 3$; the reason being that X_t, for $t = 3$, is a *proper subset* of $\overline{\operatorname{Lim}}_{t\to3} X_t = \underline{\operatorname{Lim}}_{t\to3} X_t = \operatorname{Lim}_{t\to3} X_t$. Continuity at $t = 3$ would imply that $X_3 = \operatorname{Lim}_{t\to3} X_t$.

*$\delta_\epsilon > 0$ such that, when t lies in $V_{\delta_\epsilon}^*(\tau)$, the set C lies in $U_\epsilon(X_t)$. Let D be the divisor of all sets B, and U the union of all sets C. Then $K = \overline{\lim_{t\to\tau}} X_t$ is the set D, and $L = \underline{\lim_{t\to\tau}} X_t$ is the set $\bar{U}$.**

With the understanding that a void set is closed, we may assert, even independently of the hypotheses of Theorem 4, that $\overline{\lim_{t\to\tau}} X_t$ and $\underline{\lim_{t\to\tau}} X_t$ are always closed sets.

We now record a kind of uniformity property of $\underline{\lim_{t\to\tau}} X_t$.

THEOREM 5. *Under the hypotheses of Theorem 2, there exists, corresponding to any $\epsilon > 0$, a $\delta_\epsilon > 0$ such that, if t lies in $V_{\delta_\epsilon}^*(\tau)$, and if p is an arbitrary point of $\underline{\lim_{t\to\tau}} X_t$, the divisor set $X_t \cdot U_\epsilon(p)$ is not void.*

Properties of the limit sets of a function $X_t = f(t)$, and continuity properties of the function, may be stated in terms of the "displacement" of sets. Let $d(A, B)$ denote the distance between the sets A and B in a Euclidean space; $d(A, B)$ is the greatest lower bound (minimum) of $d(a, b)$ as the points a, b range independently over A and B. We may define:

$\pi(A, B)$ = the least upper bound (maximum) of $d(A, b)$ as b ranges over B.

$\pi^\circ(A, B)$ = the greater of the numbers $\pi(A, B)$, $\pi(B, A)$, or their common value.

These displacements were introduced, purely as definitions, by Hausdorff.†

It is not difficult to establish the following theorem:

THEOREM 6. *In the function $X_t = f(t)$, let the X_t be closed sets, and let their union in G be bounded. Let τ be a proper limit point of the range T. If $X_t = f(t)$ is upper semi-continuous at $t = \tau$, then $\lim_{t\to\tau} \pi(X_\tau, X_t) = 0$, and conversely. If $X_t = f(t)$ is lower semi-continuous at $t = \tau$, then $\lim_{t\to\tau} \pi(X_t, X_\tau) = 0$, and conversely. Hence, for $f(t)$ to be continuous at $t = \tau$, it is necessary and sufficient that $\lim_{t\to\tau} \pi^\circ(X_\tau, X_t) = 0$.*

It is to be observed that π and π°, as here defined, are real-number functions of familiar type, and that "lim" denotes the ordinary limit of real-

* For definition of $\bar{U}$ cf. Section 2.

† F. Hausdorff, *Grundzüge der Mengenlehre.*

variable analysis. The equation "lim" $= 0$ implies, of course, that the limit exists.

5. *Concerning Sequences of Functions.* The following theorem is established by a kind of argument which is very familiar in analysis. The proof will be omitted here:

THEOREM 7. *Every monotone descending sequence of closed sets converges to its divisor set. Given the sequence of closed sets* $[X_i]$, *with* $X_i \supseteq X_{i+1}$, *we may write:* $Dv[X_i] = \operatorname{Lim}_{i\to\infty} X_i$.*

THEOREM 8. *Let* τ *be a proper limit point of* $T = \{t\}$. *Let each function of the denumerable sequence* $X_t^{(n)} = f^{(n)}(t)$, $(n = 1, 2, 3, \cdots)$, *be upper semi-continuous at* $t = \tau$, *and let* $X_t^{(n)} \supseteq X_t^{(n+1)}$. *Finally, let* $X_t = f(t) = \operatorname{Lim}_{n\to\infty} X_t^{(n)} = \operatorname{Lim}_{n\to\infty} f^{(n)}(t)$. *Then* $X_t = f(t)$ *is upper semi-continuous at* $t = \tau$.†

Proof: Understanding that a void set is a subset of every set, we note that the theorem is obviously valid for the case $K = \overline{\operatorname{Lim}}_{t\to\tau} X_t = 0$.

Let $K \neq 0$, and let p be any point of K. For any $\epsilon > 0$, and any $\delta > 0$, there exists in $V_\delta^*(\tau)$ a point t' such that $U_\epsilon(p)$ contains a point of $X_{t'}$. But $X_{t'} = \operatorname{Lim}_{n\to\infty} X_{t'}{}^{(n)}$. Hence for n sufficiently great, and therefore, in view of $X_{t'}{}^{(n)} \supseteq X_{t'}{}^{(n+1)}$, for any n, there exists in $V_\delta^*(\tau)$ a point t' such that $U_\epsilon(p)$ contains a point of $X_{t'}{}^{(n)}$. It follows that p is, for every n, a point of $\overline{\operatorname{Lim}}_{t\to\tau} X_t^{(n)}$; and therewith, in view of the upper semi-continuity of $X_t^{(n)}$, a point of $X_\tau^{(n)}$; and consequently‡ a point of $X_\tau = \operatorname{Lim}_{n\to\infty} X_\tau^{(n)}$. Thus $X_\tau \supseteq \overline{\operatorname{Lim}}_{t\to\tau} X_t = K$, and $f(t)$ is upper semi-continuous at $t = \tau$.

Definition: Let $X_t^{(n)} = f^{(n)}(t)$ be a denumerable sequence of functions defined over T, and converging, in T, to the function $X_t = \operatorname{Lim} X_t^{(n)}$. We shall call this convergence "semi-uniform" if, corresponding to every $\epsilon > 0$, there exists an infinite set N_ϵ of positive integers, N_ϵ *depending in general upon* ϵ, such that, for any t in T, and for any integer n in N_ϵ, we have:

$$\pi^\circ(X_t, X_t^{(n)}) < \epsilon.$$

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* Cf. footnote, page 421.

† Cf. footnote, page 421.

‡ Since $U_\epsilon(p)$ contains at least one point, p, of $X_\tau^{(n)}$ for every n, we see that p is a point of $\overline{\operatorname{Lim}}_{n\to\infty} X_\tau^{(n)}$, and therewith a point of $\operatorname{Lim}_{n\to\infty} X_\tau^{(n)}$. Cf. footnote page 421.

The convergence is "uniform" if, corresponding to every $\epsilon > 0$, there exists an index n_ϵ such that, for $n > n_\epsilon$, and for any t in T, $\pi^\circ(X_t, X_t^{(n)}) < \epsilon$.

THEOREM 9. *Let τ be a proper limit point of the range T. Let $X_t^{(n)} = f^{(n)}(t)$ be a denumerable sequence of functions converging semi-uniformly, in some partial range $Q = V_\delta(\tau)$, to the function $X_t = f(t)$. Then if each function $f^{(n)}(t)$ is continuous at $t = \tau$, so is the function $X_t = f(t)$.*

Proof: Let $[t_i]$ be any denumerable sequence of points of Q converging to τ. Let ϵ be any positive number. There exists an infinite set N_ϵ of indices, such that, for every index n in N_ϵ, and for every i, we have $\pi^\circ(X_{t_i}, X_{t_i}^{(n)}) < \epsilon$. Thus, for every n in N_ϵ, and for every i, we may write: (1) $X_{t_i} \subseteq U_\epsilon(X_{t_i}^{(n)})$, and (2) $X_{t_i}^{(n)} \subseteq U_\epsilon(X_{t_i})$.

Denoting by $\bar{U}_\epsilon(A)$, as usual, the union of $U_\epsilon(A)$ and the set of its limit points, we may conclude from (1) that, for every n in N_ϵ, and for every i: $\overline{\lim_{i\to\infty}} X_{t_i} \subseteq \bar{U}_\epsilon(\overline{\lim_{i\to\infty}} X_{t_i}^{(n)})$, and * therefore $\overline{\lim_{i\to\infty}} X_{t_i} \subseteq \bar{U}_\epsilon(X_\tau^{(n)})$. To establish this conclusion, suppose that $\overline{\lim_{i\to\infty}} X_{t_i}$ contains a point p such that the distance $d(p, X_\tau^{(n)}) = \epsilon + \zeta$, with $\zeta > 0$. Since $X_\tau^{(n)} = \lim_{i\to\infty} X_{t_i}^{(n)}$ we have only to make i greater than a definite integer i_0 to insure that $d(p, X_{t_i}^{(n)}) > \epsilon + \zeta/2$. But, for an infinity of indices i, all greater than i_0, the set X_{t_i} will have points in common with $U_{\zeta/4}(p)$, since p is a point of $\overline{\lim_{i\to\infty}} X_{t_i}$. Let j be one of these indices, and let q be a point of the divisor set $X_{t_j} \cdot U_{\zeta/4}(p)$. Then, clearly, $d(q, X_{t_j}^{(n)}) > \epsilon + \zeta/4$. Thus (1) is contradicted for $i = j$.

An analogous argument derives from (2) the formula $\overline{\lim_{i\to\infty}} X_{t_i}^{(n)} \subseteq \bar{U}_\epsilon(\overline{\lim_{i\to\infty}} X_{t_i})$, and therewith the formula $X_\tau^{(n)} \subseteq \bar{U}_\epsilon(\overline{\lim_{i\to\infty}} X_{t_i})$, valid for every integer n in N_ϵ.

Recalling that $X_\tau = \lim_{n\to\infty} X_\tau^{(n)}$, and allowing n to approach infinity, within N_ϵ, we find that, for any $\lambda > \epsilon$, certainly: (3) $\overline{\lim_{i\to\infty}} X_{t_i} \subseteq \bar{U}_\lambda(X_\tau)$, and (4) $X_\tau \subseteq \bar{U}_\lambda(\overline{\lim_{i\to\infty}} X_{t_i})$. Since ϵ is arbitrarily small, so is λ, and therefore we must have: $X_\tau = \overline{\lim_{i\to\infty}} X_{t_i}$. By theorem 3, it follows that $X_\tau = \lim_{t\to\tau} X_t$, and that $X_t = f(t)$ is continuous at $t = \tau$.

* By hypothesis, $\overline{\lim_{i\to\infty}} X_{t_i}^{(n)} = \lim_{i\to\infty} X_{t_i}^{(n)} = X_\tau^{(n)}$.

Corollary: Let τ be a proper limit point of the range T. Let $X_t^{(n)} = f^{(n)}(t)$ be a denumerable sequence of functions converging uniformly, in T, to $X_t = f(t)$. Then, if each function $f^{(n)}(t)$ is continuous at $t = \tau$, so is $f(t)$.

6. *Upper Semi-Continuous Transformation of Bounded Continua.*

THEOREM 10. *Let T be a bounded continuum in the Euclidean space E. In the Euclidean space G, let each X_t be a continuum, and let the union $U\{X_t\}$ of all the X_t sets be bounded. Then, if the function $X_t = f(t)$ is upper semi-continuous in T, the union set $U\{X_t\}$ is a continuum.*

We first show that $U\{X_t\}$ is a connected set. If this is not the case, there exist two sets U_1 and U_2 such that $U = U_1 + U_2$ and $U_1\bar{U}_2 + U_2\bar{U}_1 = 0$. Since U is bounded, we have $d(U_1, U_2) > 0$. If t is any point of T, its image, X_t, will lie wholly in U_1 or wholly in U_2, each set X_t being a bounded continuum. Let T_1, T_2 be those subsets of T whose images lie in U_1, U_2 respectively. We see that $T_1\bar{T}_2 + T_2\bar{T}_1 = 0$. For, plainly, T_1 and T_2 can have no common point, and no point of either can be a limit point of the other. If the point t' of T_1 is a limit point of T_2, there is a denumerable sequence $[t_i]$ in T_2 converging to t'. Since $X_t = f(t)$ is upper semi-continuous, there exists, corresponding to any $\epsilon > 0$, an integer n_ϵ, such that, when $i > n_\epsilon$, we have X_{t_i} lying wholly in $U_\epsilon(X_{t'})$. This is, however, impossible; $X_{t'}$ lies in U_1, while X_{t_i} lies in U_2, and $d(U_1, U_2) > 0$. Since, obviously, $T = T_1 + T_2$, the relation $T_1\bar{T}_2 + T_2\bar{T}_1 = 0$ contradicts the hypothesis that T is a continuum.

To show that $U = U\{X_t\}$ is closed, suppose that it has an improper limit point q in G. Then q is not a point of any X_t.

Let $[q_i]$ be any denumerable sequence of different points in U converging, in the sense of aggregate theory, to the point q. Let $[t_i]$ be any denumerable sequence of points of T such that q_i is a point of X_{t_i} as we may evidently do. It is plain that there are an infinity of *different* points in the sequence $[t_i]$; for otherwise an infinity of points of $[q_i]$ would lie on one set X_{t_i}, so that, X_{t_i} being closed, the limit point q of the convergent sequence $[q_i]$ would lie in X_{t_i}, and therewith in U, contrary to hypothesis. Since T is bounded, the sequence $[t_i]$ will have at least one limit point τ, lying in the closed set T. Consider the closed set X_τ. In view of the fact that q is not a point of X_τ, let $d(q, X_\tau) = h > 0$.

Since $X_t = f(t)$ is upper semi-continuous in T, so that $X_\tau \supseteq \overline{\lim_{t \to \tau}} X_t$,

there exists in $[t_i]$ an infinite * subsequence $[s_i]$, $s_i = t_{n_i}$, such that, for $i > i_0$, where i_0 is a determinate integer, we have † $X_{s_i} \subseteq U_{h/2}(X_\tau)$, and ‡ consequently $d(q_{n_i}, q) > h/2$. But this is a clear contradiction, q being the limit point of the convergent sequence $[q_i]$.

The set $U\{X_t\}$ has, therefore, no improper limit point, and is closed. Since it is both closed and connected, it is a continuum.

7. *Semi-Continuous Functions on Regional or Perfect Ranges.* We proceed to obtain another generalization of an important theorem of analysis. We shall understand, in what follows, that $X_t = f(t)$ is "semi-continuous" over, or in, the range T if it is upper semi-continuous at each point of T, or else lower semi-continuous at each point of T.

THEOREM 11. *Let the range T be bounded, and let it be either a region or a perfect set. If the function $X_t = f(t)$ is semi-continuous in T, it is at most point-wise discontinuous in T.*

To establish this theorem, we shall make use of two auxiliary real-number functions, and their properties as given in two lemmas. The functions $\delta_\epsilon(\tau)$ and $\delta_\epsilon^*(\tau)$ are defined as follows:

(a) Let τ be a point of T such that, for some point τ' of T, X_τ does not lie wholly in $\bar{U}_\epsilon(X_{\tau'})$, respectively $X_{\tau'}$ does not lie wholly in $\bar{U}_\epsilon(X_\tau)$. Then $\delta_\epsilon(\tau)$, respt. $\delta_\epsilon^*(\tau)$, is the least upper bound (maximum) of those real numbers δ having the property that, if τ^* is any point of $V_\delta(\tau)$, $X_\tau \subseteq \bar{U}_\epsilon(X_{\tau^*})$, respt $X_{\tau^*} \subseteq \bar{U}_\epsilon(X_\tau)$.

(b) Let τ be a point of T such that, for every point τ' of T, the set X_τ lies wholly in $\bar{U}_\epsilon(X_{\tau'})$, respt. the set $X_{\tau'}$ lies wholly in $\bar{U}_\epsilon(X_\tau)$. Then $\delta_\epsilon(\tau)$, respt. $\delta_\epsilon^*(\tau)$, has the value k, where k is a fixed positive number, independent of τ, which is greater than the diameter § of T.

If $\delta_\epsilon(\tau)$, respt. $\delta_\epsilon^*(\tau)$, is upper semi-continuous in the ordinary sense at $\tau = \tau^*$, we may write:

$$\overline{\lim_{\tau \to \tau^*}} \delta_\epsilon(\tau) \leqq \delta_\epsilon(\tau^*), \text{ respt. } \overline{\lim_{\tau \to \tau^*}} \delta_\epsilon^*(\tau) \leqq \delta_\epsilon^*(\tau^*)$$

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* Any subsequence which converges to τ.

† Cf. Theorem 6.

‡ Note that q_i is a point of X_{t_i}, and q_{n_i} is a point of $X_{s_i} = X_{t_{n_i}}$. The sequence $[q_{n_i}]$ is a subsequence of $[q_i]$.

§ The diameter of T is the maximum of the numbers $d(t_1, t_2)$ for all pairs of points t_1, t_2 of T.

where $\overline{\lim_{\tau \to \tau^*}} \delta_\epsilon(\tau)$, respt. $\overline{\lim_{\tau \to \tau^*}} \delta_\epsilon^*(\tau)$, denotes the minimum (greatest lower bound), as $\lambda \to 0$, of the maximum (least upper bound) of $\delta_\epsilon(\tau)$, respt. $\delta_\epsilon^*(\tau)$, in $V_\lambda(\tau^*)$.

LEMMA 1. *If $X_t = f(t)$ is lower semi-continuous, respt. upper semi-continuous, in T, the function $\delta_\epsilon(\tau)$, respt. $\delta_\epsilon^*(\tau)$, is everywhere upper semi-continuous in T, in the ordinary sense.*

Proof: Each function is clearly upper semi-continuous wherever it has its maximum value k. We shall prove the Lemma for other points of T in the case of $\delta_\epsilon(\tau)$. A precisely similar argument may be used in the case of $\delta_\epsilon^*(\tau)$.

Assuming that $X_t = f(t)$ is lower semi-continuous in T, let τ^* be a point of T where $\delta_\epsilon(\tau^*) < k$. Suppose, for argument, that $\delta_\epsilon(\tau)$ is not upper semi-continuous at $\tau = \tau^*$. Then, of course:

$$\overline{\lim_{\tau \to \tau^*}} \delta_\epsilon(\tau) = \mu + \delta_\epsilon(\tau^*), \text{ with } \mu > 0.$$

Let $\theta = \delta_\epsilon(\tau^*) + \mu/2$, $\phi = \delta_\epsilon(\tau^*) + \mu/4$, $\rho_i = (\epsilon/10)^i$, $\sigma_i = \epsilon + \rho_i$.

There exist: a denumerable sequence of positive numbers, converging steadily to zero: $\zeta_1 > \zeta_2 > \zeta_3 > \cdots$; and a denumerable sequence of points of T, converging to τ^*: $\tau_1, \tau_2, \tau_3, \cdots$; such that the following relations hold:

(1) τ_i in $V_{\zeta_i}(\tau^*)$, (2) $\delta_\epsilon(\tau_i) > \theta$, (3) $X_{\tau^*} \subset U_{\rho_i}(X_{\tau_i})$,

the postulated lower semi-continuity of $X_t = f(t)$ giving (3), in view of Theorem 6.

By definition of $\delta_\epsilon(\tau_i)$, we may assert in view of (2) that, for every i, if τ is any point of $V_\theta(\tau_i)$, we have $X_{\tau_i} \subseteq \bar{U}_\epsilon(X_\tau)$; and therefore, in view of relation (3) above, $X_{\tau^*} \subset U_{\sigma_i}(X_\tau)$.

But τ_i converges to τ^*. Hence, for i sufficiently great, we shall have $V_\phi(\tau^*) \subset V_\theta(\tau_i)$. Thus, if τ is any point of $V_\phi(\tau^*)$, we shall have, for every i, $X_{\tau^*} \subset U_{\sigma_i}(X_\tau)$. This implies that for any point τ of $V_\phi(\tau^*)$, we shall have $X_{\tau^*} \subseteq \bar{U}_\epsilon(X_\tau)$, a result contradicting the definitional property of the function $\delta_\epsilon(\tau)$, in that $\phi > \delta_\epsilon(\tau^*)$.

LEMMA 2. *If $X_t = f(t)$ is lower semi-continuous, respt. upper semi-continuous in T, and if T is a bounded region or a bounded perfect set, then $\delta_\epsilon(\tau)$, respt. $\delta_\epsilon^*(\tau)$, is at most point-wise discontinuous in T.*

This Lemma results immediately from Lemma 1 and a standard theorem of analysis. Compare the formulation of the theorem in question by C. Caratheodory, *Vorlesungen über reelle Funktionen,* p. 144, No. 142: "Jede halbstetige, im Kleinen beschränkte Funktion ist punktiert unstetig, falls ihr Definitionsbereich perfekt ist, oder offen, oder der Durchschnitt einer perfekten und einer offenen Punktmenge."

Proof of Theorem 11: We may argue concurrently the two cases involved by considering at the same time the two point-wise discontinuous functions $\delta_\epsilon(\tau)$ and $\delta_\epsilon^*(\tau)$. First statements made will apply to the case in which $X_t = f(t)$ is lower semi-continuous in T; alternative statements to the case in which $X_t = f(t)$ is upper semi-continuous.

Let $X_t = f(t)$ be everywhere lower semi-continuous, respt. everywhere upper semi-continuous, in T. Let $0 < \epsilon^* < 1/2$, and consider the denumerable sequence of positive numbers $[\epsilon_i]$, with $\epsilon_i = (\epsilon^*/2)^i$. Let τ_1 in T be any point of *continuity* of $\delta_{\epsilon_1}(\tau)$, respt. of $\delta_{\epsilon_1}^*(\tau)$. In view of Lemma 2, if $\bar{\tau}$ is any point of T, and if $\epsilon > 0$, the point τ_1 may be chosen in $V_\epsilon(\bar{\tau})$.

Let $\rho_1 > 0$ be chosen arbitrarily small, and so as to satisfy the requirements:

(a) $\rho_1 < \frac{1}{4}\delta_{\epsilon_1}(\tau_1)$, respt. $\rho_1 < \frac{1}{4}\delta_{\epsilon_1}^*(\tau_1)$

(b) If τ' and τ'' are points of $V_{\rho_1}(\tau_1)$, then

$$|\delta_{\epsilon_1}(\tau') - \delta_{\epsilon_1}(\tau'')| < \tfrac{1}{4}\delta_{\epsilon_1}(\tau_1), \text{ respt. } |\delta_{\epsilon_1}^*(\tau') - \delta_{\epsilon_1}^*(\tau'')| < \tfrac{1}{4}\delta_{\epsilon_1}^*(\tau_1)$$

the absolute value of a real number a being denoted by $|a|$.

In $V_{\rho_1}(\tau_1)$, choose a point τ_2 at which $\delta_{\epsilon_2}(\tau)$, respt. $\delta_{\epsilon_2}^*(\tau)$, is *continuous.* If T is a region, choose τ_2 so that it lies in the same component of T with τ_1. This can be done, by virtue of Lemma 2.

Choose $\rho_2 > 0$ so that:

(a) $\rho_2 < \frac{1}{4}\delta_{\epsilon_2}(\tau_2)$, respt. $\rho_2 < \frac{1}{4}\delta_{\epsilon_2}^*(\tau_2)$

(b) If τ' and τ'' are points of $V_{\rho_2}(\tau_2)$, then

$$|\delta_{\epsilon_2}(\tau') - \delta_{\epsilon_2}(\tau'')| < \tfrac{1}{4}\delta_{\epsilon_2}(\tau_2), \text{ respt. } |\delta_{\epsilon_2}^*(\tau') - \delta_{\epsilon_2}^*(\tau'')| < \tfrac{1}{4}\delta_{\epsilon_2}^*(\tau_2)$$

(c) $\rho_2 < D_1/4$, where D_1 denotes the diameter of the set $V_{\rho_1}(\tau_1)$

(d) $V_{\rho_2}(\tau_2) \subset V_{\rho_1}(\tau_1)$

(e) If T is a region, $V_{\rho_2}(\tau_2)$ lies in the component of T containing τ_2.

This procedure is to be continued indefinitely. In $V_{\rho_{i-1}}(\tau_{i-1})$, choose the point τ_i at which $\delta_{\epsilon_i}(\tau)$, respt. $\delta_{\epsilon_i}{}^*(\tau)$, is continuous; if T is a region, τ_i is to be chosen in the component of T containing τ_{i-1}. Cf. Lemma 2. Choose $\rho_i > 0$ in such manner that:

(a) $\rho_i < \frac{1}{4}\delta_{\epsilon_i}(\tau_i)$, respt. $\rho_i < \frac{1}{4}\delta_{\epsilon_i}{}^*(\tau_i)$

(b) If τ' and τ'' are points of $V_{\rho_i}(\tau_i)$, then

$$|\delta_{\epsilon_i}(\tau') - \delta_{\epsilon_i}(\tau'')| < \tfrac{1}{4}\delta_{\epsilon_i}(\tau_i), \text{ respt. } |\delta_{\epsilon_i}{}^*(\tau') - \delta_{\epsilon_i}{}^*(\tau'')| < \tfrac{1}{4}\delta_{\epsilon_i}{}^*(\tau_i)$$

(c) $\rho_i < \frac{1}{4}D_{i-1}$, where D_i is the diameter of the set $V_{\rho_i}(\tau_i)$

(d) $V_{\rho_i}(\tau_i) \subset V_{\rho_{i-1}}(\tau_{i-1})$

(e) If T is a region, $V_{\rho_i}(\tau_i)$ lies in the component of T containing τ_i.

By the construction outlined, it is evident that in every case the sequence of points $[\tau_i]$ converges in the usual sense to a definite limit point τ^* in T.

If τ' and τ'' are any two points of $V_{\rho_i}(\tau_i)$, we have simultaneously the two relations:

$$(1) \qquad X_{\tau'} \subset \bar{U}_{\epsilon_i}(X_{\tau''}), \qquad (2) \qquad X_{\tau''} \subset \bar{U}_{\epsilon_i}(X_{\tau'})$$

which are both valid whether $X_t = f(t)$ is lower or upper semi-continuous. To demonstrate these relations, we note that in view of (a) the distance $d(\tau', \tau'')$ is less than $\frac{1}{2}\delta_{\epsilon_i}(\tau_i)$ respt. less than $\frac{1}{2}\delta_{\epsilon_i}{}^*(\tau_i)$; and that in view of (b) we have:

$$|\delta_{\epsilon_i}(\tau_i) - \delta_{\epsilon_i}(\tau')| < \tfrac{1}{4}\delta_{\epsilon_i}(\tau_i), \text{ respt. } |\delta_{\epsilon_i}{}^*(\tau_i) - \delta_{\epsilon_i}{}^*(\tau')| < \tfrac{1}{4}\delta_{\epsilon_i}{}^*(\tau_i)$$

as well as

$$|\delta_{\epsilon_i}(\tau_i) - \delta_{\epsilon_i}(\tau'')| < \tfrac{1}{4}\delta_{\epsilon_i}(\tau_i), \text{ respt. } |\delta_{\epsilon_i}{}^*(\tau_i) - \delta_{\epsilon_i}{}^*(\tau'')| < \tfrac{1}{4}\delta_{\epsilon_i}{}^*(\tau_i)$$

It follows that both of the numbers $\delta_{\epsilon_i}(\tau')$, $\delta_{\epsilon_i}(\tau'')$, respt. both of the numbers $\delta_{\epsilon_i}{}^*(\tau')$, $\delta_{\epsilon_i}{}^*(\tau'')$, are greater than $\frac{3}{4}\delta_{\epsilon_i}(\tau_i)$, respt. $\frac{3}{4}\delta_{\epsilon_i}{}^*(\tau_i)$. Consequently, both of the numbers $\delta_{\epsilon_i}(\tau')$, $\delta_{\epsilon_i}(\tau'')$, respt. both of the numbers $\delta_{\epsilon_i}{}^*(\tau')$, $\delta_{\epsilon_i}{}^*(\tau'')$, are greater than $d(\tau', \tau'')$. By definition of the functions $\delta_{\epsilon_i}(\tau)$, $\delta_{\epsilon_i}{}^*(\tau)$, the simultaneous relations (1), (2) immediately follow.

Corresponding to any positive integer i, we can find a positive number q, such that $V_q(\tau^*) \subset V_{\rho_i}(\tau_i)$, τ^* being the limit point of the sequence $[\tau_i]$. Thus we can choose q so that, if τ is any point of $V_q(\tau^*)$, the set X_τ, respt. the set $X_\tau{}^*$, will lie in $\bar{U}_{\epsilon_i}(X_\tau{}^*)$, respt. in $\bar{U}_{\epsilon_i}(X_\tau)$.

Given any $\epsilon > 0$, we can find an integer i_0, such that, for $i > i_0$, we have $\epsilon_i < \epsilon$. Hence, for any $\epsilon > 0$, we can find $q > 0$ such that, when τ lies in $V_q(\tau^*)$, we have $X_\tau \subset U_\epsilon(X_\tau{}^*)$, respt. $X_\tau{}^* \subset U_\epsilon(X_\tau)$, the dash over the U being dropped.

For the lower semi-continuous function $X_t = f(t)$, respt. for the upper semi-continuous function, we have found, in the arbitrarily small neighborhood $V_{\rho_1}(\tau_1)$, of the point τ_1, a point τ^* at which $X_\tau{}^* \supseteq \overline{\underset{\tau \to \tau^*}{\mathrm{Lim}}}\, X_\tau$, respt. $X_\tau{}^* \subseteq \underset{\overline{\tau \to \tau^*}}{\mathrm{Lim}}\, X_\tau$; and therewith, in each case, a point τ^* of continuity of the function.*

But τ_1 was chosen arbitrarily from a set of points everywhere dense in T. Hence T contains an everywhere dense set of points at each of which the function $X_t = f(t)$ is continuous, and Theorem 11 is established.

The term " region " has been used to include the case of an open interval, or the union of a set of open intervals, when T lies in one-way space.

The supracontinuous functions employed by W. A. Wilson, in Sections 26-28 of paper cited,† have herewith been shown to be at most point-wise discontinuous. For Theorem 11 applies, *à fortiori,* to functions which are everywhere supracontinuous, or everywhere infracontinuous, in a bounded regional or perfect range T.

It is not difficult to generalize somewhat the hypotheses of Theorem 11. So strong a limitation as that requiring T to be a bounded set is not necessary.

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* From the relation $X_\tau \subseteq U_\epsilon(X_{\tau^*})$, just established, we conclude, on the basis of Theorem 4, that $X_{\tau^*} \supseteq \overline{\underset{t \to \tau^*}{\mathrm{Lim}}}\, X_\tau$. In like manner, Theorem 4 converts the relation $X_{\tau^*} \subseteq U_\epsilon(X_\tau)$ into $X_{\tau^*} \subseteq \underset{\overline{t \to \tau^*}}{\mathrm{Lim}}\, X_\tau$.

† Cf. footnote §, page 419.

The Theory of Group-Reduced Distributions.

By J. Howard Redfield.

In view of the similarity which will be admitted to hold between the subject matters of the Theory of Finite Groups and of Combinatory Analysis, it is somewhat surprising to find that in their literatures the two branches have proceeded on their separate ways without developing their interrelationship, and with scarcely any reference to one another beyond the use by each of certain very elementary results of the other.

In the present paper the connection established between the two branches takes the form of associating every permutation group with a certain symmetric function. By means of these functions and certain differential operations closely allied with those of Hammond and MacMahon, a theory is developed which is in a sense an extension of MacMahon's theory of Chess-Board Diagrams (*Combinatory Analysis,* vol. I, p. 224 ff.). From the point of view here adopted, this theory of MacMahon may be regarded as a special one in which the groups involved are exclusively such as are direct products of symmetric groups on distinct sets of elements, and in which we deal with collections of objects divided into kinds and freely permutable within each kind. In the extended theory the groups may be of any type whatever. We are thus enabled to treat a much wider variety of combinatorial problems; some of them have previously been solved by special devices, but even as to these we gain generality of outlook. One class of such problems, striking because having a convenient geometrical interpretation, though otherwise not more important than some others, has to do with configurations based on the regular polygons and polyhedra; the groups of rotations involved clearly lie outside the domain of the chess-board diagram type of group.

At the same time nearly every problem which we are able to solve suggests others which do not yield to the methods here developed, but which seemingly should nevertheless be amenable to treatment by methods further perfected along the same lines. Belief is therefore warranted that this borderland region would repay much more extensive investigation. While for the moment Combinatory Analysis will be seen to be the chief beneficiary, some of the results obtained below are not without interest from the point of view of Group Theory, and both branches may be expected to profit by future work in this field.

1. *Preliminary Definitions.* We consider an *assemblage* $A(n, q)$ *of* nq *elements,* divided into q *ranges* $S_1, S_2, \cdots, S_q$ of n elements each. We form therewith correspondences, here termed *range-correspondences,* of type defined by a representation wherein the ranges are written in q lines so as to form a rectangular array of n columns. Range-correspondences will be regarded as identical if their arrays consist of identical columns, regardless of the order of the columns. Thus the number of possible range-correspondences will be $(n!)^{q-1}$.

With each range S_r $(r = 1, 2, \cdots, q)$ we associate a specified permutation group (its *range-group*) G_r, of degree n and order m_r, operating on the elements of S_r. The combined operations of these q groups will determine a group Γ of degree $(n!)^{q-1}$ and order $m_1 m_2 \cdots m_q$, whose elements will be the $(n!)^{q-1}$ range-correspondences, which Γ will interchange among themselves. In general, Γ will not be transitive, that is, the combined groups $G_1, G_2, \cdots, G_q$ will not transform every range-correspondence into every other. Such transformation will however be possible within each of a number θ of closed classes, here termed *group-reduced distributions* (being the transitive constituents of Γ), into which the $(n!)^{q-1}$ range-correspondences are parcelled by the given groups $G_1, G_2, \cdots, G_q$.

By the symbol $N(G_r; p_1^{\pi_1} p_2^{\pi_2} \cdots)$ we denote the number of operations, contained in the group G_r, which exhibit, when written in the usual cycle notation, π_1 cycles of order p_1, π_2 cycles of order p_2, etc.

A partition such as $p_1^{\pi_1} p_2^{\pi_2} \cdots$ which specifies the orders of the different cycles in the symbol of a group operation, will be called the *cycle-partition* of the operation; thus the operation $abc \cdot de \cdot fg \cdot h$ is said to be of cycle-partition $32^2 1$.

THEOREM: *The number* θ *of group-reduced distributions of an assemblage* $A(n, q)$ *determined by the range-groups* $G_1, G_2, \cdots, G_q$ *whose respective orders are* $m_1, m_2, \cdots, m_q$, *has the expression*

$$\theta = \frac{\Sigma\{(p_1^{\pi_1} p_2^{\pi_2} \cdots \pi_1! \pi_2! \cdots)^{q-1} \cdot N(G_1; p_1^{\pi_1} p_2^{\pi_2} \cdots) \cdot N(G_2; p_1^{\pi_1} p_2^{\pi_2} \cdots) \cdots N(G_q; p_1^{\pi_1} p_2^{\pi_2} \cdots)\}}{m_1 m_2 \cdots m_q}$$

in which under Σ *there is a term for every partition* $p_1^{\pi_1} p_2^{\pi_2} \cdots$ *of the degree* n *common to all the range-groups.*

Proof: Since the range-groups operate on distinct sets (ranges) of elements, operations of different range-groups will be commutative. The direct product of the range-groups is a group of degree qn, simply isomorphic with

the group Γ defined above; Γ may be called the *group of range-correspondences.*

Let γ be an operation of Γ, and let $J(\gamma)$ denote the number of range-correspondences (i. e. elements of Γ) which are not altered by γ. The sum $\Sigma[J(\gamma)]$, taken for all the $m_1 m_2 \cdots m_q$ operations of Γ, gives an enumeration in which every range-correspondence β_1 is counted as many times as there are operations in Γ which do not alter β_1. Let $K(\beta_1)$ denote the number of operations of Γ which thus leave β_1 unaltered.

If β_1 is unaltered by γ_1 and γ_2, it is unaltered by γ_1^{-1}, γ_2^{-1}, $\gamma_1\gamma_2$ and $\gamma_2\gamma_1$. Hence those operations of Γ which do not alter β_1 form a sub-group U_1, whose order $K(\beta_1)$ is a divisor of $m_1 m_2 \cdots m_q$.

Let $1, u_2, u_3, \cdots, u_{K(\beta_1)}$ be the operations of U_1, and let v_2 be any operation of Γ which is not in U_1. Then v_2 will change β_1 into a range-correspondence β_2, different from β_1 but belonging to the same group-reduced distribution. It is clear that β_1 will be changed into β_2 by every operation of the set $v_2, u_2v_2, u_3v_2, \cdots, u_{K(\beta_1)}v_2$, and that β_2 will be unaltered by every operation of the set $1, v_2^{-1}u_2v_2, v_2^{-1}u_3v_2, \cdots, v_2^{-1}u_{K(\beta_1)}v_2^2$, which latter form a sub-group U_2 of Γ, isomorphic with U_1. Using other operations v_3, v_4, etc. we can in like manner reach every range-correspondence of the group-reduced distribution to which β_1 belongs, at the same time accounting for all the operations of Γ. The reasoning is identical with that employed in proving that the order of a group is a multiple of the order of each of its subgroups (e. g Miller, Blichfeldt, and Dickson, *Finite Groups,* p. 22), and establishes that the group-reduced distribution δ of which β_1 is a member contains altogether $m_1 m_2 \cdots m_q / K(\beta_1)$ range-correspondences. But every range-correspondence in δ is counted $K(\beta_1)$ times in the sum $\Sigma[J(\gamma)]$, and therefore δ itself is counted $K(\beta_1) \times [m_1 m_2 \cdots m_q / K(\beta_1)] = m_1 m_2 \cdots m_q$ times. Since this number $m_1 m_2 \cdots m_q$ is independent of $K(\beta_1)$ and the same for every one of the θ group-reduced distributions, it follows that $\theta = \Sigma[J(\gamma)] / m_1 m_2 \cdots m_q$.

It remains to evaluate $\Sigma[J(\gamma)]$, and we observe in the first place that $J(\gamma) = 0$ for every γ which is determined by a set of range-group operations which are not all of the same cycle-partition. For if to any range-correspondence β_1 we apply an operation (not the identity) of G_1 (say), the range-correspondence will be altered, and, to restore it, we must evidently apply to each of the ranges $S_2, S_3, \cdots, S_q$, an operation of the same cycle-partition with the operation first performed on S_1; while if the first operation is the identity of G_1, all the others must also be the identities of their range-groups.

Supposing, then, that γ is determined by the q operations $g_1, g_2, \cdots, g_q$, all of cycle-partition $p_1^{r_1} p_2^{r_2} \cdots$; let these operations be written in the cycle

notation, in q lines, so that they form a rectangular array with cycles of equal order over one another. The elements forming columns will then determine a range-correspondence β_1 which is not altered by γ. Attending now to the first line of the array, the operation g_1 exhibits π_1 cycles of order p_1; each of these can be rewritten so as to begin with any one of its p_1 symbols, and the π_1 cycles can be permuted as wholes in $\pi_1!$ ways. We thus get $p_1^{\pi_1}\pi_1!$ variations, and when the same is done with the π_2 cycles of order p_2, and the rest, we have $p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots$ equivalent notations for the same operation g_1. The matching of the orders of cycles in the q rows of the array is not disturbed, but instead of the single range-correspondence β_1 we now have $p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots$ range-correspondences (including β_1) which are unaltered by γ. Similar variations can be effected on each of the remaining $q-1$ rows of the array. But if *all* the rows are varied, every range-correspondence occurring will be repeated $p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots$ times, since permutations of whole columns of an array do not alter the range-correspondence which it represents. Evidently we shall obtain all the distinct range-correspondences each once only if we keep one row fixed as a reference base and vary independently the remaining $q-1$ rows. We thus find that the total number $J(\gamma)$ of distinct range correspondences which are unaltered by γ, is equal to $(p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots)^{q-1}$. Hence

$$\theta = \frac{\Sigma[J(\gamma)]}{m_1 m_2 \cdots m_q} = \frac{\Sigma[(p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots)^{q-1}]}{m_1 m_2 \cdots m_q},$$

the summation Σ covering every operation γ of Γ which is determined by a set of operations of $G_1, G_2, \cdots, G_q$ which are all of the same cycle-partition. The theorem as stated in terms of $N(G_r; p_1^{\pi_1}p_2^{\pi_2}\cdots)$ immediately follows.

2. *The Differential Operations* Ω *and* ϑ. In the foregoing developments, for greater generality, no use has been made of the theory of Symmetric Functions, which at this point it becomes convenient to introduce. We shall have occasion to consider only such symmetric functions as are rational and integral, and the term is to be understood throughout with this restriction.

For the details of the theory reference is made to MacMahon's *Combinatory Analysis* (vol. I, sec. I, chap. I), of which the notation will in general be here followed. For convenience however we may here recall that we deal with the symmetric functions of a set of ν indeterminates or symbolic quantities $\alpha_1, \alpha_2, \cdots, \alpha_\nu$. The number ν is not specified, since ordinarily we make no use of expressions which are dependent on its value.

A partition (of a number w) enclosed in round brackets, as $(p_1p_2p_3\cdots p_k)$, the k parts being not necessarily unequal, denotes the sum $\sum[\alpha_{i_1}{}^{p_1}\alpha_{i_2}{}^{p_2}\cdots\alpha_{i_k}{}^{p_k}]$ of all the *different* terms which can be formed by taking for $i_1, i_2, \cdots, i_k$ every possible ordered set of k different suffixes selected from the integers $1, 2, 3, \cdots, \nu$. Specially, s_w is written for (w), a_w for (1^w), and h_w for the sum $\sum[(l_1{}^{\lambda_1}l_2{}^{\lambda_2}\cdots)]$ taken for every partition $l_1{}^{\lambda_1}l_2{}^{\lambda_2}\cdots$ of w; thus

$$h_4 = (4) + (31) + (2^2) + (21^2) + (1^4).$$

MacMahon (loc. cit., Vol. I, sec. I, chap. I) gives formulae for expressing the s-, a-, and h-functions in terms of one another. These can be derived, by the method of undetermined coefficients or otherwise, from the following generating function identities:

$$\text{(A)} \qquad \begin{cases} (1-\alpha_1x)(1-\alpha_2x)\cdots(1-\alpha_\nu x) \\ = 1 - a_1x + a_2x^2 - a_3x^3 + \cdots \\ = 1/(1 + h_1x + h_2x^2 + h_3x^3 + \cdots) \\ = \exp(-s_1x - \tfrac{1}{2}s_2x^2 - \tfrac{1}{3}s_3x^3 - \cdots) \end{cases}$$

Of the Hammond differential operators, MacMahon makes d_w correspond to s_w, and D_w to a_w. Here however we shall use the equally valid correspondence of δ_w to s_w, and D_w to h_w, since by so doing we can replace certain alternating $+$ and $-$ signs in our expressions by $+$ signs throughout.

It is convenient to take as fundamental the operator δ_w, which MacMahon (loc. cit., Vol. I, p. 36) defines thus:

$$\text{(B)} \qquad \delta_w = \frac{\partial}{\partial h_w} + h_1\frac{\partial}{\partial h_{w+1}} + h_2\frac{\partial}{\partial h_{w+2}} + \cdots$$

We proceed to show that δ_w is equivalent to $w(\partial/\partial s_w)$, a very simple relation of which MacMahon makes no use.

Equating logarithms of reciprocals of the third and fourth expressions of (A), we have

$$\text{(C)} \qquad \log(1 + h_1x + h_2x^2 + \cdots) = s_1x + \tfrac{1}{2}s_2x^2 + \tfrac{1}{3}s_3x^3 + \cdots .$$

Operating with δ_w on the right-hand member of (C), and with the equivalent

$$\frac{\partial}{\partial h_w} + h_1\frac{\partial}{\partial h_{w+1}} + h_2\frac{\partial}{\partial h_{w+2}} + \cdots$$

on the left-hand member, we obtain

$$(D) \qquad \frac{x^w + h_1 x^{w+1} + h_2 x^{w+2} + \cdots}{1 + h_1 x + h_2 x^2 + \cdots}$$

$$= \delta_w s_1 \cdot x + \tfrac{1}{2}\delta_w s_2 \cdot x^2 + \cdots + (1/w)\ \delta_w s_w \cdot x^w + \cdots .$$

The left-hand member of (D) reduces to x^w, and by equating terms containing like powers of x we find that $\delta_w s_v = 0$ if $w \neq v$, while $\delta_w s_w = w$, so that

$$\delta_w = w\ \frac{\partial}{\partial s_w} .$$

It follows then that $\delta_{\lambda_1}{}^{l_1}\delta_{\lambda_2}{}^{l_2}\cdots s_{\mu_1}{}^{m_1} s_{\mu_2}{}^{m_2}\cdots = 0$ whenever $\lambda_1{}^{l_1}\lambda_2{}^{l_2}\cdots$ and $\mu_1{}^{m_1}\mu_2{}^{m_2}\cdots$ are different partitions of the same number, while $\delta_{\lambda_1}{}^{l_1}\delta_{\lambda_2}{}^{l_2}\cdots s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots = \lambda_1{}^{l_1}\lambda_2{}^{l_2}\cdots l_1!\, l_2!\cdots$.

We introduce a new connective symbol of operation $\otimes$, and write by definition $s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots \otimes s_{\mu_1}{}^{m_1}s_{\mu_2}{}^{m_2}\cdots$ for $\delta_{\lambda_1}{}^{l_1}\delta_{\lambda_2}{}^{l_2}\cdots s_{\mu_1}{}^{m_1}s_{\mu_2}{}^{m_2}\cdots$, with the further defining relation $(A+B)\otimes C = (A\otimes C) + (B\otimes C)$, expressing the distributive law with respect to addition. MacMahon's $D_{\lambda_1}{}^{l_1}D_{\lambda_2}{}^{l_2}\cdots h_{\mu_1}{}^{m_1}h_{\mu_2}{}^{m_2}\cdots$ or $D_{\mu_1}{}^{m_1}D_{\mu_2}{}^{m_2}\cdots h_{\lambda_1}{}^{l_1}h_{\lambda_2}{}^{l_2}\cdots$ is then readily seen to be equivalent to $h_{\lambda_1}{}^{l_1}h_{\lambda_2}{}^{l_2}\cdots \otimes h_{\mu_1}{}^{m_1}h_{\mu_2}{}^{m_2}\cdots$.

We extend the definition of $\otimes$ so as to write $s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots \otimes s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots \otimes \cdots \otimes s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots$ (to q operands) $= (\lambda_1{}^{l_1}\lambda_2{}^{l_2}\cdots l_1!\, l_2!\cdots)^{q-1}$, while the result is zero if not all the operands exhibit the same partition $\lambda_1{}^{l_1}\lambda_2{}^{l_2}\cdots$. Thus extended, the operator $\otimes$ is commutative and distributive as to addition, but the result cannot be decomposed into simpler $\otimes$-operations, and the notion of associativeness has no relevance.

Another new symbol of operation $\otimes\!\!\!\!\bigcirc$ is defined by $s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots \otimes\!\!\!\!\bigcirc\, s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots = (s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots \otimes s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots)\, s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots$, with $(A+B)\otimes\!\!\!\!\bigcirc\, C = (A\otimes\!\!\!\!\bigcirc\, C) + (B\otimes\!\!\!\!\bigcirc\, C)$; $s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots \otimes\!\!\!\!\bigcirc\, s_{\mu_1}{}^{m_1}s_{\mu_2}{}^{m_2}\cdots$ vanishing as in the case of $\otimes$ when $\lambda_1{}^{l_1}\lambda_2{}^{l_2}\cdots$ and $\mu_1{}^{m_1}\mu_2{}^{m_2}\cdots$ are different partitions. Both $\otimes$ and $\otimes\!\!\!\!\bigcirc$ connect only symmetric functions of equal weights, but whereas $\otimes$ yields a pure number, $\otimes\!\!\!\!\bigcirc$ yields a symmetric function of the same weight with the operands. $\otimes\!\!\!\!\bigcirc$ is evidently commutative, distributive as to addition, and associative, so that we can write unambiguously without parentheses $A_1 \otimes\!\!\!\!\bigcirc\, A_2 \otimes\!\!\!\!\bigcirc\, A_3 \otimes\!\!\!\!\bigcirc \cdots \otimes\!\!\!\!\bigcirc\, A_q$. It is also readily verified that $(A\otimes\!\!\!\!\bigcirc\, B)\otimes C = A\otimes(B\otimes\!\!\!\!\bigcirc\, C) = A\otimes B\otimes C$.

3. *The Group-Reduction Function* of any permutation group G of degree n and order m, is defined to be the symmetric function

$$\mathrm{Grf}(G) = (1/m)\,\Sigma[N(G; p_1{}^{r_1}p_2{}^{r_2}\cdots)\ s_{p_1}{}^{r_1}s_{p_2}{}^{r_2}\cdots],$$

in which N has the same meaning as in § 1 and Σ covers every partition $p_1^{\pi_1}p_2^{\pi_2}\cdots$ of the weight n. It does not completely characterize its group, since the same G. R. F. may be shared by two or more distinct permutation groups (isomorphic or conformal).

The G. R. F.'s of the symmetric and alternating groups of degree n are, in MacMahon's notation, respectively h_n and $h_n + a_n$. For, the symmetric group of degree n contains all the possible operations of all cycle-partitions which can be formed with n elements. Considering the cycle-partition $p_1^{\pi_1}p_2^{\pi_2}\cdots$, we first write in the cycle notation a typical operation, putting blanks — in the places to be occupied by the n elements; thus for the cycle-partition 32^2 of 7 we should write $(---)(--)(--)$. The blanks can now be filled by the n elements $\alpha_1, \alpha_2, \cdots, \alpha_n$ in any permutation; this gives $n!$ operation symbols, which are however not all distinct. For each cycle of p_r elements can be made to begin with any one of its elements, and if there are π_r such cycles, these can be permuted in $\pi_r!$ ways, and this can be done independently for each value of r; all these varieties are different notations for a single operation. Therefore the number of distinct operations of cycle-partition $p_1^{\pi_1}p_2^{\pi_2}\cdots$ in the symmetric group of degree n is

$$n!/(p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots), \quad \text{and the G. R. F. is}$$

$$(1/n!)\,\Sigma\left(\frac{n!}{p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots}\; s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots\right)$$

$$= \Sigma\left(\frac{s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots}{p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots}\right).$$

But this is the expression of h_n in terms of s-functions given by MacMahon (loc. cit., Vol. I, p. 7).—The alternating group of degree n contains the even permutations of the corresponding symmetric group, and its G. R. F. is therefore

$$(2/n!)\,\Sigma\left(\frac{n!}{p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots}\; s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots\right),$$

in which the summation includes only those cycle-partitions which correspond to even permutations, that is, those having an even number of even parts. Now the expression of a_n in terms of s-functions (MacMahon, loc. cit., Vol. I, p. 6) differs from the expression for h_n only in that the terms exhibiting partitions with an odd number of even parts, have the negative sign. In the sum $h_n + a_n$ these terms cancel out, while the terms exhibiting partitions with an even number of even parts are doubled. If the factor 2 thus arising is carried outside the Σ sign, the non-vanishing coefficients under the Σ remain

as in the expression of h_n, and we find that the G. R. F. of the alternating group, as given above, is equal to $h_n + a_n$.

The G. R. F.'s of the cyclic groups are of considerable theoretical as well as practical importance. By $\mathrm{Cyc}(p_1^{\pi_1}p_2^{\pi_2}\cdots)$ we shall denote the G. R. F. of a cyclic group generated by an operation whose cycle-partition is $p_1^{\pi_1}p_2^{\pi_2}\cdots$, a partition of the degree w of the group; its order is of course the least common multiple of $p_1, p_2, \cdots$. Such a group is transitive if and only if it is generated by an operation whose cycle-partition is the one-part partition w, and in that case it is easily verified that

$$\mathrm{Cyc}(w) = (1/w)[s_1^w + \phi(\alpha)s_\alpha^{w/\alpha} + \phi(\beta)s_\beta^{w/\beta} + \cdots + \phi(w)s_w],$$

in which $1, \alpha, \beta, \cdots, w$ are all the divisors of w, and $\phi(\zeta)$ denotes the number of positive integers, including 1, less than and prime to ζ. Thus $\mathrm{Cyc}(6) = (1/6)(s_1^6 + s_2^3 + 2s_3^2 + 2s_6)$. In the G. R. F. of an intransitive cyclic group, the coefficients are the same as in the G. R. F. of the transitive cyclic group of the same order, but the literal parts of the terms are s-function products exhibiting the cycle-partitions of the generating operation and its powers. Thus $\mathrm{Cyc}(3^2) = (1/6)(s_1^7 + s_2^2s_1^3 + 2s_3s_1^4 + 2s_3s_2^2)$.

The G. R. F. of a dihedral group derived from a transitive cyclic group of degree w will be denoted by $\mathrm{Dih}(w)$. For odd and even degrees we have respectively

$$\mathrm{Dih}(2k+1) = \tfrac{1}{2}\mathrm{Cyc}\ (2k+1) + \tfrac{1}{2}s_2^k s_1;$$
$$\mathrm{Dih}(2k) = \tfrac{1}{2}\mathrm{Cyc}(2k) + \tfrac{1}{4}s_2^k + \tfrac{1}{4}s_2^{k-1}s_1^2.$$

It is easily seen that the algebraic product of two or more G. R. F.'s is the G. R. F. of the direct product of the groups belonging to the factors, written on distinct sets of symbols.

The enumerating expression for group-reduced distributions can now be rewritten in the form

$$\theta = \mathrm{Grf}(G_1) \mathbin{\overline{\times}} \mathrm{Grf}(G_2) \mathbin{\overline{\times}} \cdots \mathbin{\overline{\times}} \mathrm{Grf}(G_q).$$

This results immediately from the Theorem of § 1 and the definition of $\overline{\times}$ in § 2.

For every group G of degree n, $h_n \mathbin{\overline{\times}} \mathrm{Grf}(G) = 1$, and $h_n \mathbin{\overline{\times}} Grf(G) = \mathrm{Grf}(G)$. For in $\mathrm{Grf}(G)$, let the coefficient of $s_{p_1^{\pi_1}}s_{p_2^{\pi_2}}\cdots$ be $A(p_1^{\pi_1}p_2^{\pi_2}\cdots)$. In h_n the coefficient of $s_{p_1^{\pi_1}}s_{p_2^{\pi_2}}\cdots$ is $1/(p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots)$. (MacMahon, *loc. cit.*, vol. I, p. 7). Then $A(p_1^{\pi_1}p_2^{\pi_2}\cdots)s_{p_1^{\pi_1}}s_{p_2^{\pi_2}}\cdots \mathbin{\overline{\times}} s_{p_1^{\pi_1}}s_{p_2^{\pi_2}}\cdots/p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots = A(p_1^{\pi_1}p_2^{\pi_2}\cdots)$, by § 2. Thus $h_n \mathbin{\overline{\times}} \mathrm{Grf}(G) = \Sigma[A(p_1^{\pi_1}p_2^{\pi_2}\cdots)]$ taken for all partitions $p_1^{\pi_1}p_2^{\pi_2}\cdots$ of n. But this is the sum of the co-

efficients of $\mathrm{Grf}(G)$, and we know from the definition of a G. R. F. (§ 3) that this sum is equal to unity.—Also, since

$$A(p_1^{\pi_1}p_2^{\pi_2}\cdots)s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots \mathbin{\wr} s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots / p_1^{\pi_1}p_2^{\pi_2}\cdots \pi_1!\,\pi_2!\cdots = A(p_1^{\pi_1}p_2^{\pi_2}\cdots)s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots,$$

it follows immediately that the $\wr$-product of h_n and any symmetric function (including any G. R. F.) of degree n, merely reproduces the original symmetric function.

If $h_{n-k}h_1^k \mathbin{\wr} \mathrm{Grf}(G) = 1$, then G is at least k-ply transitive, and conversely. For let the two ranges of the assemblage be S_1 and S_2, of which let S_1 be associated with the group G, and S_2 with a group H whose G. R. F. is $h_{n-k}h_1^k$; then H is the direct product of the symmetric group on $n-k$ symbols, by the group of identity on k other symbols. As elements of S_2 we may take $n-k$ similar symbols α, permutable among themselves in all possible ways, together with k distinct symbols $\beta_1, \beta_2, \cdots, \beta_k$, none of which are displaced by any operation of H. Let the elements of S_1 be $\gamma_1, \gamma_2, \cdots, \gamma_n$. If now G is at least k-ply transitive, the k elements of S_1 which are paired with $\beta_1, \beta_2, \cdots, \beta_k$ in any range-correspondence, can by some operation contained in G be replaced by any specified set of k elements of S_1, and this moreover in any one of the $k!$ possible correspondences. Thereafter the $n-k$ elements α can be permuted in any specified way by a suitable operation contained in H. Thus from any range correspondence we can pass to any other by operations of G and H, so that there is only one group-reduced distribution, and $h_{n-k}h_1^k \mathbin{\wr} \mathrm{Grf}(G) = 1$. But if G is not k-ply transitive, then not every set of k elements of S_1 can be brought into correspondence with $\beta_1, \beta_2, \cdots, \beta_k$, and there will be more than one group-reduced distribution; $h_{n-k}h_1^k \mathbin{\wr} \mathrm{Grf}(G) > 1$. Thus the proposition and its converse are established.

If $h_{n-1}h_1 \mathbin{\wr} \mathrm{Grf}(G) = t$, then G has t transitive constituents. For let S_1 as before be associated with the group G, and S_2 with a group H whose G. R. F. is $h_{n-1}h_1$; then we may take, as elements of S_2, $n-1$ similar symbols α, with a distinct symbol β. Starting with any range-correspondence in which an element γ of S_1 corresponds to β, the various operations of G will replace γ by those elements of S_1, and those alone, which belong to the same transitive constituent of G as does γ; while the operations of H, which affect only the α's, will unite into one group-reduced distribution all the range-correspondences in which a particular element of S_1 is paired with β. Thus there will be a group-reduced distribution for each of the t transitive constituents of G, and $h_{n-1}h_1 \mathbin{\wr} \mathrm{Grf}(G) = t$.

It is convenient, so far as it can be done significantly and unambiguously, to speak of a G. R. F. as transitive, cyclic, of degree n, etc., when the corresponding group has the property in question.

Examples of Application: (i) Required the number of distinct configurations which can be obtained by placing a solid node • at each of four vertices of a cube, and a hollow node ° at each of the four remaining vertices, configurations differing only in orientation not being regarded as distinct.

The group of rotations of the cube is simply isomorphic with the symmetric group on 4 symbols, and when written on 8 symbols representing the vertices, has the G. R. F.

$$V = (1/24)(s_1^8 + 9s_2^4 + 8s_3^2s_1^2 + 6s_4^2).$$

The group of all node-permutations which do not alter colors of nodes is the direct product of two symmetric groups of degree 4 on distinct sets of symbols, and has as G. R. F.

$$h_4^2 = [(1/24)(s_1^4 + 6s_2s_1^2 + 3s_2^2 + 8s_3s_1 + 6s_4)]^2$$

$$= (1/576)(s_1^8 + 12s_2s_1^6 + 42s_2^2s_1^4 + 36s_2^3s_1^2 + 9s_2^4 + 16s_3s_1^5 + 96s_3s_2s_1^3 + 48s_3s_2^2s_1 + 64s_3^2s_1^2 + 12s_4s_1^4 + 72s_4s_2s_1^2 + 36s_4s_2^2 + 96s_4s_3s_1 36s_4^2).$$

The computation of $h_4^2 \otimes V$ is as follows:

$1 \times 1 \times 1^8 \times 8!$	=	40320
$9 \times 9 \times 2^4 \times 4!$	=	31104
$64 \times 8 \times 3^2 1^2 \times (2!)^2$	=	18432
$36 \times 6 \times 4^2 \times 2!$	=	6912
Product of orders: $576 \times 24 = 13824$	)	96768
Number of configurations: $h_4^2 \otimes V =$		7

Be it noted that only those cycle-partitions need be considered which are represented in *all* the G. R. F.'s entering as operands. This "orthogonality" of the s-function products with respect to $\otimes$ and $\otimes$ greatly simplifies the computations. In the present case, however, and usually when one of the operands is a monomial product of h-functions, it would be simpler to use the D-operators and to compute D_4^2V by the rules which MacMahon gives (*loc. cit.*, p. 43).

The actual configurations, shown below, cannot be determined by the methods of the present theory, but must be found, as in all other cases, by detailed consideration of the groups involved, and this may of course be very laborious, except in simple cases, or where special devices are available.

(1) (2) (3) (4) (5) (6) (7)

In connection with the present example we may note without proof certain other simple results obtainable.

Thus if in V we substitute $x^r + y^r$ for every s_r, we obtain the polynomial

$$x^8 + x^7y + 3x^6y^2 + 3x^5y^3 + 7x^4y^4 + 3x^3y^5 + 3x^2y^6 + xy^7 + y^8,$$

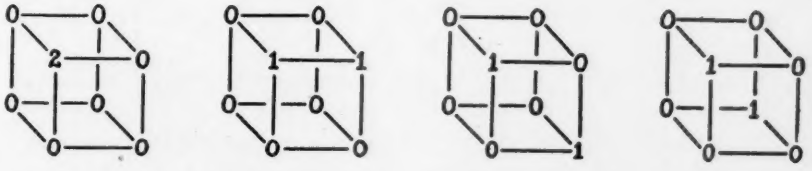
in which the coefficient of $x^t y^{8-t}$ enumerates the distinct configurations possible with t nodes • and $8 - t$ nodes °.

The sum of the coefficients in the above expression is 23, which is the total number of configurations when the numbers of nodes of the two colors are not specified. This enumeration is also effected by substituting 2 for every s_r in V. Similarly if k colors are available we substitute k for every s_r; thus with 3 colors there are $(1/24)(3^8 + 9.3^4 + 8.3^4 + 6.3^2) = 333$ possible configurations.

If in V we put $1/(1 - x^r)$ for every s_r, we obtain the infinite series

$$1 + x + 4x^2 + 7x^3 + 21x^4 + 37x^5 + \cdots,$$

in which the coefficient of x^t enumerates the distinct configurations obtained by placing a zero or a positive integer at every vertex of the cube, subject to the condition that the sum of the 8 numbers is always t. For $t = 2$, the 4 configurations are



If in V we put 2 for every s_{2k} and 0 for every s_{2k+1}, we enumerate the configurations in which it is possible to change the color of every node into

the opposite color by a suitable rotation of the cube. The number is found to be $(1/24)(0 + 9.2^4 + 0 + 6.2^2) = 7$, and it will be seen that in fact all the 7 configurations shown above have this property. In other analogous cases the like is not always true; thus for the 12 vertices of the icosahedron there are 24 distinct configurations with 6 nodes of each color, only 16 of which interchange colors by rotation.

(ii) Let n points be placed at the vertices of a regular n-gon. Starting from any one of these points, a closed route is traced, consisting of straight segments running from each point to the next in any order. Required the number of distinct configurations obtained, when no account is taken of differences of orientation or of any particular starting point of a route.

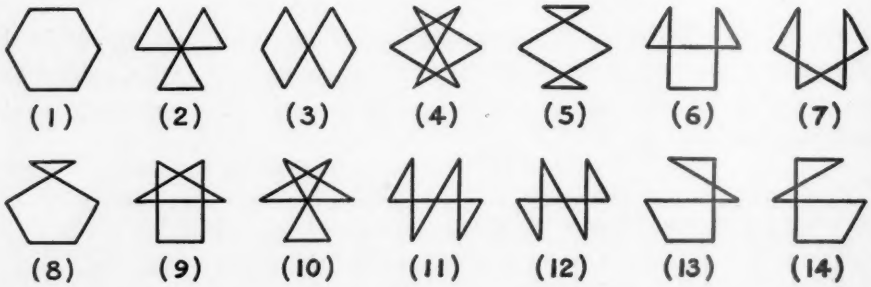
If we distinguish (a) between routes traversed in opposite senses and (b) between a configuration and its reflection, we shall have as range-groups two cyclic groups of degree and order n, and the number of configurations will be $\mathrm{Cyc}(n) \wr \mathrm{Cyc}(n)$. Thus if $n = 6$ we shall have

$$(1/6)(s_1^6 + s_2^3 + 2s_3^2 + 2s_6) \wr (1/6)(s_1^6 + s_2^3 + 2s_3^2 + 2s_6)$$
$$= (1/36)(6! + 2^3.3! + 2^2.3^2.2! + 2^2.6) = 24.$$

If we disregard one only of the distinctions (a) and (b) above, the number of configurations will be $\mathrm{Cyc}(n) \wr \mathrm{Dih}(n)$. For $n = 6$ we shall have

$$(1/6)(s_1^6 + s_2^3 + 2s_3^2 + 2s_6) \wr (1/12)(s_1^6 + 3s_2^2s_1^2 + 4s_2^3 + 2s_3^2 + 2s_6)$$
$$= (1/72)(6! + 4.2^3.3! + 2^2.3^2.2! + 2^2.6) = 14.$$

In the diagrams shown below, distinction (a) is disregarded.



If we specify the direction of travel, as by arrow heads, we obtain two diagrams from each of the above; but these are not distinct in the cases of (3), (4), (11), and (12), since a reversal of direction is there equivalent to a rotation through 180° about an axis through the center of the figure and

perpendicular to its plane. The remaining 10 pairs are distinct and increase the total to $14 + 10 = 24$, as computed.

If we abolish *both* distinctions (a) and (b), the enumeration is given by $\mathrm{Dih}(n) \otimes \mathrm{Dih}(n)$; for $n = 6$ this is

$$(1/12)(s_1^6 + 3s_2^2s_1^2 + 4s_2^3 + 2s_3^2 + 2s_6) \otimes (1/12)(s_1^6 + 3s_2^2s_1^2 + 4s_2^3 + 2s_3^2 + 2s_6)$$
$$= (1/144)(6! + 3^2.2^2.1^2.2!^2 + 4^2.2^3.3! + 2^2.3^2.2! + 2^2.6) = 12.$$

And in fact diagram (12) is the reflection of (11), and (14) of (13); thus the number of distinct configurations is now reduced to $14 - 2 = 12$.

§ 4. *Decomposition of* $\wr$*-Products.*

THEOREM: *If* $\mathrm{Grf}(G_1) \otimes \mathrm{Grf}(G_2) \otimes \cdots \otimes \mathrm{Grf}(G_q) = \theta$, *then* $\mathrm{Grf}(G_1) \wr \mathrm{Grf}(G_2) \wr \cdots \wr \mathrm{Grf}(G_q)$ *can be expressed as the sum* $\Phi_1 + \Phi_2 + \cdots + \Phi_\theta$ *of* θ *G. R. F.'s, which can be associated in one-one correspondence with the* θ *group-reduced distributions* $\delta_1, \delta_2, \cdots, \delta_\theta$ *determined by* $G_1, G_2, \cdots, G_q$, *in such wise that if* Φ_σ *is the correspondent of* δ_σ, *and* β *is any range-correspondence of* δ_σ, *then those operations of any range-group* G_r *which leave* β *unaltered when they are performed in combination with suitable similar operations of the remaining G's, form a sub-group whose G. R. F. is* Φ_σ. For different β's belonging to the same δ these sub-groups are similar, but not necessarily identical.

Proof: In the Proof of the Theorem of § 1, we saw that

$$\theta = \Sigma[J(\gamma)]/m_1m_2 \cdots m_q,$$

and in § 3 we saw that this expression could be written in the form $\mathrm{Grf}(G_1) \otimes \mathrm{Grf}(G_2) \otimes \cdots \otimes \mathrm{Grf}(G_q)$. We also saw (§ 1, Proof) that whenever $J(\gamma)$ does not vanish, γ is determined by a set of operations of $G_1, G_2, \cdots, G_q$ which are all of the same cycle-partition, say $p_1^{\tau_1}p_2^{\tau_2}\cdots$. If now we attach to each $J(\gamma)$ a literal multiplier $s_{p_1^{\tau_1}s_{p_2^{\tau_2}}\cdots}$ embodying the cycle-partition $p_1^{\tau_1}p_2^{\tau_2}\cdots$ characterizing γ, it is evident that the modified expression $\Sigma[J(\gamma) \cdot s_{p_1^{\tau_1}s_{p_2^{\tau_2}}\cdots}]/m_1m_2\cdots m_q$ will be equal to $\mathrm{Grf}(G_1) \wr \mathrm{Grf}(G_2) \wr \cdots \wr \mathrm{Grf}(G_q)$.

But $\Sigma[J(\gamma)]$ contains a unit, and therefore $\Sigma[J(\gamma) \cdot s_{p_1^{\tau_1}s_{p_2^{\tau_2}}\cdots}]$ contains a summand $s_{p_1^{\tau_1}s_{p_2^{\tau_2}}\cdots}$, for every instance in which a range-correspondence β_1 is associated with an operation γ, of cycle-partition $p_1^{\tau_1}p_2^{\tau_2}\cdots$, which leaves β_1 unaltered. If we attend to the summands $s_{p_1^{\tau_1}s_{p_2^{\tau_2}}\cdots}$ associated with a single range-correspondence β_1, we see, since the corresponding operations form a group U_1 of order $K(\beta_1)$, that all these $s_{p_1^{\tau_1}s_{p_2^{\tau_2}}\cdots}$ taken together give

an expression equal to $K(\beta_1)$ times the G. R. F. of U_1. The groups U_2, U_3, $\cdots$, etc., belonging to the range-correspondences β_2, β_3, $\cdots$ which are in the same group-reduced distribution with β_1, are all similar to U_1 and so have the same G. R. F. If we extend our summation to include this whole group-reduced distribution, which contains $m_1 m_2 \cdots m_q / K(\beta_1)$ range-correspondences, the result is

$$\frac{m_1 m_2 \cdots m_q}{K(\beta_1)} \cdot K(\beta_1)\Phi_\sigma = m_1 m_2 \cdots m_q \, \Phi_\sigma,$$

in which Φ_σ is the G. R. F. belonging indifferently to U_1, U_2, U_3, $\cdots$, etc. Finally, if we include all the θ group-reduced distributions, we obtain

$$m_1 m_2 \cdots m_q (\Phi_1 + \Phi_2 + \cdots + \Phi_\theta),$$

in which Φ_1, Φ_2, $\cdots$, Φ_θ is the complete set of G. R. F.'s of which Φ_σ is a typical one. But since we have now used all the summands $s_{p_1}{}^{\tau_1} s_{p_2}{}^{\tau_2} \cdots$, the above expression must be equal to $\sum[J(\gamma) \cdot s_{p_1}{}^{\tau_1} s_{p_2}{}^{\tau_2} \cdots]$, whence:

$$\begin{aligned} \Phi_1 + \Phi_2 + \cdots + \Phi_\theta &= \sum[J(\gamma) \cdot s_{p_1}{}^{\tau_1} s_{p_2}{}^{\tau_2} \cdots] / m_1 m_2 \cdots m_q \\ &= \mathrm{Grf}(G_1) \wr \mathrm{Grf}(G_2) \wr \cdots \wr \mathrm{Grf}(G_q). \end{aligned}$$

The decomposition is thus proved to be theoretically possible in all cases. But for actually effecting it for the $\wr$-product of given G. R. F.'s, no general method has yet been found. Moreover, if the $\wr$-product involves operands which can belong to two or more distinct groups, the decomposition may not be unique.

Thus, the G. R. F. $(1/4)(s_1{}^6 + 3s_2{}^2 s_1{}^2)$ is shared by the two isomorphic but distinct groups

$$\begin{aligned} Q_1 &= [1;\ ab{\cdot}cd{\cdot}e{\cdot}f;\ ab{\cdot}c{\cdot}d{\cdot}ef;\ a{\cdot}b{\cdot}cd{\cdot}ef]; \\ Q_2 &= [1;\ ab{\cdot}cd{\cdot}e{\cdot}f;\ ac{\cdot}bd{\cdot}e{\cdot}f;\ ad{\cdot}bc{\cdot}e{\cdot}f]. \end{aligned}$$

Both are intransitive, with 3 transitive constituents, as shown by

$$h_5 h_1 \wr (1/4)(s_1{}^6 + 3s_2{}^2 s_1{}^2) = 3;$$

but the transitive constituents are of degrees 2 2 2 and 4 1 1 respectively. $h_5 h_1 \wr (1/4)(s_1{}^6 + 3s_2{}^2 s_1{}^2) = (1/4)(6s_1{}^6 + 6s_2{}^2 s_1{}^2)$, which has the two decompositions $3[(1/2)(s_1{}^6 + s_2{}^2 s_1{}^2)]$ and $2[(1/4)(s_1{}^6 + 3s_2{}^2 s_1{}^2)] + s_1{}^6$, the first corresponding to Q_1 and the second to Q_2.

Reverting to the example on cube configurations (§ 3, i), we have

$$\begin{aligned} &h_4{}^2 \wr (1/24)(s_1{}^8 + 9s_2{}^4 + 8s_3{}^2 s_1{}^2 + 6s_4{}^2) \\ &\quad = (1/24)(70s_1{}^8 + 54s_2{}^4 + 32s_3{}^2 s_1{}^2 + 12s_4{}^2), \end{aligned}$$

which decomposes into

$$
\begin{aligned}
&(1) && (1/12)(s_1^8 + 3s_2^4 + 8s_3^2 s_1^2)\\
&(2) && + (1/4)(s_1^8 + s_2^4 + 2s_4^2)\\
&(3) && + (1/4)(s_1^8 + 3s_2^4)\\
&(4) && + (1/3)(s_1^8 + 2s_3^2 s_1^2)\\
&(5) \text{ and } (6) && + 2[(1/2)(s_1^8 + s_2^4)]\\
&(7) && + s_1^8.
\end{aligned}
$$

where the numbers to the left indicate the correspondence of the component G. R. F.'s with the seven diagrams shown in § 3, i. The reader will easily verify that each of them is the G. R. F. of the group of rotations which does not alter colors of nodes in the corresponding cube configuration. The decomposition

$$(1/4)(s_1^8 + s_2^4 + 2s_4^2) + 2[(1/3)(s_1^8 + 2s_3^2 s_1^2)] + 4[(1/2)(s_1^8 + s_2^4)]$$

is algebraically possible and involves none but admissible common sub-groups, yet it is not valid; this will give some notion of the difficulty of the general problem of decomposition.

5. *Special Cases of Decomposition.* The decomposition of $\wr$-products into G. R. F.'s can be accomplished uniquely in two important special classes of cases:

CASE 1: THEOREM: *When the operands $F_1, F_2, \cdots, F_q$ are* all *of them G. R. F.'s of the form $h_{\lambda_1}{}^{l_1} h_{\lambda_2}{}^{l_2} \cdots$, then $\Phi = F_1 \wr F_2 \wr \cdots \wr F_q$ is a sum of G. R. F.'s each of which is likewise a monomial product of h-functions, and the decomposition is found by transforming Φ, when it presents itself in terms of s-functions, into an expression in h-functions, using the formulae given by MacMahon* (loc. cit., Vol. I, p. 6).

Proof: For let the q ranges be $S_1, S_2, \cdots, S_q$, associated with groups $G_1, G_2, \cdots, G_q$, each G_r having a G. R. F. such as $h_{p_1}{}^{\pi_1} h_{p_2}{}^{\pi_2} \cdots$. We take, as elements of each S_r, p_1 symbols of each of π_1 kinds, p_2 symbols of each of π_2 kinds, etc., so that G_r admits all possible permutations among symbols of the same kind, but admits no interchange of symbols of different kinds. Let the ranges be written in a rectangular array of q rows and n columns, the elements of each range being in any arbitrary order; thus for 3 ranges of 7 elements each, the G. R. F.'s being $h_3 h_2^2$, $h_3^2 h_1$, $h_4 h_2 h_1$, we might have

$$\begin{matrix} A & A & B & A & C & B & C \\ a & b & b & a & c & b & a \\ \alpha & \gamma & \gamma & \alpha & \beta & \alpha & \beta \end{matrix}$$

Such an array symbolizes a range-correspondence, which is not altered by permutations of entire columns. But such permutations of entire columns are admitted by $G_1, G_2, \cdots, G_q$ if and only if the elements interchanged in each row are alike, which requires that the entire columns interchanged be alike; and any set of like columns admits under $G_1, G_2, \cdots, G_q$ all its possible permutations. Thus the group of admissible column-permutations for any range-correspondence, is a direct product of symmetric groups, having a G. R. F. of the form $h_{\lambda_1}{}^{l_1} h_{\lambda_2}{}^{l_2} \cdots$. In the above example, the columns $Aa\alpha$ and $Bb\alpha$ occur twice each, the others once each; so that the range-correspondence represented admits a group whose G. R. F. is $h_2{}^2 h_1{}^3$. The same being true of every range-correspondence and of every group-reduced distribution involved in the problem, it follows that the $\wr$-product of all the G. R. F.'s can be decomposed into a sum of G. R. F.'s each of which is a product of h-functions.

Since the $\wr$-product is a symmetric function, and every symmetric function has a unique expression as a linear function of h-function products, it follows that this expression, when found by the formulae mentioned above, will give the required decomposition.

Chess Board Diagrams. MacMahon's theory of Chess Board Diagrams comes under this case (loc. cit., Vol. I, sec. V, chap. II). The coefficient of $h_{\lambda_1}{}^{l_1} h_{\lambda_2}{}^{l_2} \cdots$ in the decomposition of a $\wr$-product of monomial h-function products enumerates the diagrams in which the assemblage of numbers appearing in the various diagram compartments forms the partition $\lambda_1{}^{l_1} \lambda^{l_2} \cdots$. Using three or more operands we can enumerate analogous diagrams in three or more dimensions; the theory of these, though not explicitly given as such, is contained in Vol. I, sec. II, chap. V of *Combinatory Analysis*. The example which MacMahon gives at pp. 85-86 can be looked upon as an enumeration of three-dimensional block diagrams, having 2 1 1, 2 1 1, and 2 2 units in respective rows, columns, and layers. The enumerating expression for all the diagrams is $h_2 h_1{}^2 \wr h_2 h_1{}^2 \wr h_2{}^2 = 38$. The corresponding $\wr$-product is $h_2 h_1{}^2 \wr h_2 h_1{}^2 \wr h_2{}^2$

$$= (1/2)(s_1{}^4 + s_2 s_1{}^2) \wr (1/2)(s_1{}^4 + s_2 s_1{}^2) \wr (1/4)(s_1{}^4 + 2 s_2 s_1{}^2 + s_2{}^2)$$
$$= \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot \tfrac{1}{4} \cdot \{[1 \times 1 \times 1 \times (1^4 \cdot 4!)^2] s_1{}^4 + [1 \times 1 \times 2 \times (2.1^2.2!)^2] s_2 s_1{}^2\}$$
$$= 36 s_1{}^4 + 2 s_2 s_1{}^2 = 34 s_1{}^4 + 4[(1/2)(s_1{}^2 + s_2)] s_1{}^2 = 34 h_1{}^4 + 4 h_2 h_1{}^2;$$

so that there are 34 "unitary" diagrams with one unit in each of 4 compartments, and 4 diagrams with 2 units in one compartment and one unit in each of 2 others. If in MacMahon's expressions involving the magnitudes b_μ (loc. cit.) we put h_μ for every b_μ, we obtain the expression of a $\wr$-product in terms of h-functions..

CASE 2: THEOREM: *If of the operands $F_1, F_2, \cdots, F_q$ at least one is the G. R. F. of a cyclic group, say* $\mathrm{Cyc}(\lambda_1{}^{l_1}\lambda_2{}^{l_2}\cdots)$, *then* $\Phi = F_1 \wr F_2 \wr \cdots \wr F_q$ *can be expressed as a sum of G. R. F.'s of cyclic groups, each of which is either* $\mathrm{Cyc}\,(\lambda_1{}^{l_1}\lambda_2{}^{l_2}\cdots)$ *itself, or the G. R. F. of a cyclic group generated by some power of an operation of cycle-partition* $\lambda_1{}^{l_1}\lambda_2{}^{l_2}\cdots$. *The decomposition is made as follows: From* Φ *obliterate the term* $Cs_{\mu_1}{}^{m_1}s_{\mu_2}{}^{m_2}\cdots$ *which corresponds to the operations of highest order involved, by subtracting a suitable multiple* $k\,\mathrm{Cyc}(\mu_1{}^{m_1}\mu_2{}^{m_2}\cdots)$ *of the G. R. F. of the cyclic group generated by a similar operation; treat the remainder* $\Phi - k\,\mathrm{Cyc}\,(\mu_1{}^{m_1}\mu_2{}^{m_2}\cdots)$ *in the same way; and continue the process until all the terms of* Φ *are obliterated; the G. R. F.'s thus subtracted will constitute the decomposition required.*

Proof: For every summand composing Φ is (by § 4, Theorem) the G. R. F. of a sub-group of each of the groups belonging to the F's, of which groups one, being by hypothesis cyclic, can have as sub-groups only itself and the cyclic groups generated by the powers of its own generator.

If Φ contains no terms which represent operations of higher order than does $Cs_{\mu_1}{}^{m_1}s_{\mu_2}{}^{m_2}\cdots$, then this term must be contributed by a component $\mathrm{Cyc}(\mu_1{}^{m_1}\mu_2{}^{m_2}\cdots)$ of Φ, repeated a number k of times sufficient to make up the coefficient C; and the same reasoning applies to the successive remainders left after subtraction of $k\,\mathrm{Cyc}\,(\mu_1{}^{m_1}\mu_2{}^{m_2}\cdots)$ and its analogues.

Thus $h_3h_2 \wr \mathrm{Cyc}(32)$

$$= (1/12)\,(s_1{}^5 + 4s_2s_1{}^3 + 3s_2{}^2s_1 + 2s_3s_1{}^2 + 2s_3s_2)$$
$$\wr\,(1/6)\,(s_1{}^5 + s_2s_1{}^3 + 2s_3s_1{}^2 + 2s_3s_2)$$
$$= (1/72)\,(120s_1{}^5 + 48s_2s_1{}^3 + 24s_3s_1{}^2 + 24s_3s_2)$$
$$= (1/6)\,(10s_1{}^5 + 4s_2s_1{}^3 + 2s_3s_1{}^2 + 2s_3s_2).$$

Subtraction of $\mathrm{Cyc}(32)$ or $(1/6)\,(s_1{}^5 + s_2s_1{}^3 + 2s_3s_1{}^2 + 2s_3s_2)$ leaves
$$(1/6)\,(9s_1{}^5 + 3s_2s_1{}^3);$$

from this subtracting $\mathrm{Cyc}\,(21^3)$ or $(1/2)\,(s_1{}^5 + s_2s_1{}^3)$ leaves $s_1{}^5$; from this subtracting $\mathrm{Cyc}(1^5)$ or $s_1{}^5$ leaves 0. Therefore

$$h_3h_2 \wr \mathrm{Cyc}(32) = \mathrm{Cyc}(32) + \mathrm{Cyc}(21^3) + \mathrm{Cyc}(1^5).$$

6. *Derivation of G. R. F.'s of Isomorphs.* If $\phi(\alpha_1, \alpha_2, \cdots, \alpha_n)$, a function of n arguments, preserves the same value when these arguments are permuted in its symbol by any operation of a specified group H of degree n and order μ, but changes its value when its arguments are permuted by any operation not contained in H, then $\phi(\alpha_1, \alpha_2, \cdots, \alpha_n)$ assumes $n!/\mu$ different values in all when the same set of n arguments is inserted in its symbol in all possible permutations. This is a well-known result (comp. e. g. C. Jordan, *Traité des Substitutions,* pp. 25, 26), but we may here point out that it is readily established by a simple application of the theory of Group-Reduced Distributions, by taking as one range the n arguments $\alpha_1, \alpha_2, \cdots, \alpha_n$, with G. R. F. Grf$(H)$, and as the second range the n fixed positions in the symbol of ϕ, with the group of identity on n elements as range-group, G. R. F. s_1^n. Now Grf(H) is always of the form $(1/\mu)(s_1^n + \cdots)$, the omitted terms of which are irrelevant in the $\otimes$-product $s_1^n \otimes (1/\mu)(s_1^n + \cdots) = n!/\mu$, which is the number of distinct values which ϕ can assume. It is unnecessary to think of ϕ as a particular algebraic or other function; it is merely a symbol which we agree to regard as equivalent to any similar symbol obtainable from it by any operation of H performed on its arguments.

Any permutation of the n α's determines a corresponding permutation of the $n!/\mu$ ϕ's, and since all the α's are on the same footing as to the positions which they may occupy in the symbol $\phi(—, —, \cdots, —)$, all permutations on the α's which are of a given cycle-partition P (a partition of n) will be in correspondence with permutations of the ϕ's which are of one and the same cycle-partition P' (a partition of $n!/\mu$).

The G. R. F. of any group G of degree n can be transformed, according to the correspondence so established, into the G. R. F. of a group G' of degree $n!/\mu$, isomorphic with G. In this transformation all the coefficients of Grf(G) are unaltered, and only the literal symbols $s_{\lambda_1}^{l_1} s_{\lambda_2}^{l_2} \cdots$ are changed in accordance with the correspondence of cycle-partitions. The isomorphism is not necessarily simple, and the G. R. F.'s determined by various groups H are not always distinct.

We adopt for the G. R. F. of the derived group G' the notation Grf(H) δ Grf(G). We also introduce the concept of the "powers" of a cycle-partition as follows: If P denotes any partition, P^k will denote that partition which is the cycle-partition of the k-th power u^k of any permutation-operation u which is itself of cycle-partition P. Thus if $P = 43$, we shall have

$$P = P^5 = P^7 = P^{11} = 43; \quad P^2 = P^{10} = 32^2; \quad P^3 = P^9 = 41^3;$$
$$P^4 = P^8 = 31^4; \quad P^6 = 2^2 1^3; \quad P^{12} = 1^7.$$

THEOREM: *To every operation of order ξ and cycle-partition P, in a group G of degree n, there corresponds, in the derived group G' whose G. R. F. is* $\mathrm{Grf}(H) \wr \mathrm{Grf}(G)$, *an operation of cycle-partition* $1^{A_1}\alpha^{A_\alpha}\beta^{A_\beta}\cdots\xi^{A_\xi}$, *where* $1, \alpha, \beta, \cdots, \xi$ *are the divisors of* ξ, *and the A's are the coefficients in the expression* $A_1\mathrm{Cyc}(P) + A_\alpha\mathrm{Cyc}(P^\alpha) + A_\beta\mathrm{Cyc}(P^\beta) + \cdots + A_\xi s_1^n$, *obtained by the decomposition of* $\mathrm{Grf}(H) \wr \mathrm{Cyc}(P)$ *by Case 2 of § 5 above.*

Proof: We introduce for ϕ a new notation

$$\phi\begin{pmatrix}\alpha_f & \alpha_g & \cdots & \alpha_l \\ \beta_p & \beta_q & \cdots & \beta_z\end{pmatrix},$$

in which $f, g, \cdots, l$ and $p, q, \cdots, z$ are permutations of the numbers $1, 2, \cdots, n$, and a column such as $\begin{matrix}\alpha_g \\ \beta_q\end{matrix}$ indicates that α_g occupies the qth position in the old notation $\phi(—, —, \cdots, —)$. The meaning of the new symbol is evidently not changed by permutations of whole columns; this purely notational invariance is to be distinguished from the special invariance determined by the group H operating on the upper row alone.

The α's and the β's may be regarded as two ranges S_α, S_β, with H as the group associated with S_α, while with S_β we associate a cyclic group C, generated by an operation u of order ξ and cycle-partition P, so that $\mathrm{Grf}(C) = \mathrm{Cyc}(P)$. Let η be the smallest integer such that the operation u^η, performed on the β's of a given ϕ, yields a symbol ϕ' equivalent to ϕ in virtue of the invariance under H. Then η must be either ξ or a divisor of ξ, and the operations $u, u^2, \cdots, u^\eta$ will yield successively η symbols $\phi_1, \phi_2, \cdots, \phi_{\eta-1}, \phi$ all distinct under the invariance determined by H. In the cyclic group C', into which H transforms C, the operation u will thus be represented by an operation u' of degree $n!/\mu$ and cycle-partition P', P' being a partition of $n!/\mu$ containing a part η, since one of the cycles of u' will be $(\phi\phi_1\phi_2\cdots\phi_{\eta-1})$. On the other hand the two-rowed symbol of ϕ can be read as a range-correspondence which is unaltered only by such operations as combine u^η or one of its powers with a suitable similar operation of H. But u^η generates a cyclic group whose G. R. F. is $\mathrm{Cyc}(P^\eta)$, and so $\mathrm{Grf}(H) \wr \mathrm{Grf}(C) = \mathrm{Grf}(H) \wr \mathrm{Cyc}(P)$ will have (by the Theorems of § 4 and of § 5 Case 2) a component $\mathrm{Cyc}(P^\eta)$, in correspondence with a part η in P'. Extension to the complete $\wr$-product then establishes the rule stated for finding P'.

Example: Take $\mathrm{Grf}(H) = h_3h_2$; this being of order $3!2! = 12$ will transform any G. R. F. of degree 5 into an isomorphic G. R. F. of degree

$5!/12 = 10$. We found above (§ 5 Case 2) $h_3h_2 \wr \mathrm{Cyc}(32) = \mathrm{Cyc}(32) + \mathrm{Cyc}(21^3) + \mathrm{Cyc}(1^5)$. $\mathrm{Cyc}(32)$ belongs to a group whose order 6 has the divisors 1, 2, 3, 6. We have $P = 32$, $P^2 = 31^2$, $P^3 = 21^3$, $P^6 = 1^5$; also $A_1 = A_3 = A_6 = 1$, $A_2 = 0$; whence $1^{A_1}2^{A_2}3^{A_3}6^{A_6} = 631$. Thus we obtain the cycle-partition correspondence $32 : 631$. Moreover, since a cycle-partition correspondence between two group operations implies that the same correspondence holds between like powers of those operations, we have at once the correspondences $31^2 : 3^31$, $21^3 : 2^31^4$, and $1^5 : 1^{10}$. Further correspondences are given by $h_3h_2 \wr \mathrm{Cyc}(41) = \mathrm{Cyc}(2^21) + 2\mathrm{Cyc}(1^5)$, namely $41 : 4^22$, and $2^21 : 2^41^2$, and by $h_3h_2 \wr \mathrm{Cyc}(5) = 2\mathrm{Cyc}(1^5)$, namely $5 : 5^2$. Then we have for the complete set of 7 partitions of 5:

Partitions of 5:	5	41	32	31^2	2^21	21^3	1^5
Partitions of 10:	5^2	4^22	631	3^31	2^41^2	2^31^4	1^{10}.

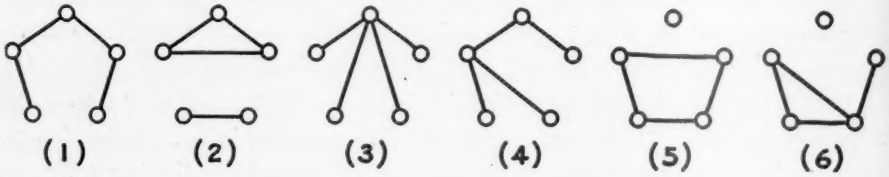
This set of correspondences serves to derive an isomorph of degree 10 from any group of degree 5. Thus from $(1/5)(s_1^5 + 4s_5) = \mathrm{Cyc}(5)$ comes $(1/5)(s_1^{10} + 4s_5^2) = \mathrm{Cyc}(5^2)$. Or taking for G the symmetric group of degree 5, we have from

$$h_5 = (1/120)(s_1^5 + 10s_2s_1^3 + 15s_2^2s_1 + 20s_3s_1^2 + 20s_3s_2 + 30s_4s_1 + 24s_5)$$

the isomorphic G. R. F.

$$h_3h_2 \wr h_5 = (1/120)(s_1^{10} + 10s_2^3s_1^4 + 15s_2^4s_1^2 + 20s_3^3s_1 + 30s_4^2s_2 + 24s_5^2 + 20s_6s_3s_1).$$

The function last written is typical of a class of G. R. F.'s $h_{n-2}h_2 \wr h_n$, of degree $n(n-1)/2$ and order $n!$, connected with the enumeration of the symmetrical aliorelative dyadic "relation-numbers" (Whitehead and Russell, *Principia Mathematica,* Vol. II, p. 301) on a field of n elements. If we represent field members by nodes o, and the holding of a typical relation by connecting lines, then there are 10 possible connecting lines for 5 nodes. If exactly 4 of these 10 are drawn, we get the configurations shown below, which are enumerated by $h_6h_4 \wr (h_3h_2 \wr h_5) = 6$.



Configurations (5) and (6), which contain isolated nodes, do not, according to the accepted definitions, represent relation-numbers on a field of *five*

elements. What we have enumerated are in fact the relation-numbers of a certain type on fields of *five or fewer* elements, and to remove the redundancy we must subtract the corresponding expression for fields of four or fewer elements, thus: $h_6h_4 \wr (h_3h_2 \wr h_5) - h_2h_4 \wr (h_2^2 \wr h_4) = 6 - 2 = 4$.

By putting every $s_r = 2$ (see § 3 ex. i) in the expressions $h_{n-2}h_2 \wr h_n$ for various values of n, we find that the total number of symmetrical aliorelative dyadic relation-numbers on fields of n elements is as follows:

Elements in Field:	1	2	3	4	5	6	7	...
Relation-Numbers:	0	1	2	7	23	122	888	...

Other species of relation-numbers can be enumerated in a similar manner. Thus for asymmetrical aliorelative dyadic relation-numbers we use the functions $h_{n-2}h_1^2 \wr h_n$: for symmetrical aliorelative triadic relation numbers the functions $h_{n-3}h_3 \wr h_n$; for aliorelative triadic relation-numbers symmetrical in two only of their arguments, the functions $h_{n-3}h_2h_1 \wr h_n$.

Sometimes the isomorphism is not simple but j-fold; in such case every derived operation will be repeated j times, and this will give a leading term js_1^n inside the parentheses of the G. R. F. When this factor j is carried outside the parentheses, the G. R. F. will appear as of the correct order, so that no exception arises to the rule for finding the expression for $\mathrm{Grf}(H) \wr \mathrm{Grf}(G)$.

Other isomorphs can be obtained by using as transforming function, instead of a G. R. F. $\mathrm{Grf}(H)$, any symmetric function which is the sum of two or more G. R. F.'s of the proper degree. Thus for enumerating the symmetrical dyadic relation numbers which are not restricted to be aliorelative, we use functions $(h_{n-2}h_2 + h_{n-1}h_1) \wr h_n$. Such isomorphs are necessarily intransitive. On the other hand, when a single G. R. F. is used as transforming function, the derived isomorph *may* be intransitive, and if it is, still other isomorphs can be got by omitting some of the transitive constituents. The general problem of actually separating an intransitive G. R. F. into transitive constituents has however not been solved, and seems to be closely connected in nature and difficulty with the decomposition of $\wr$-products into sums of G. R. F.'s.

Some important general relations are (for G and H of degree n, H of order μ): $h_n \wr \mathrm{Grf}(G) = s_1$ for every G; $(h_n + a_n) \wr \mathrm{Grf}(G) = (1/2)(s_1^2 + s_2)$ or s_1^2 according as G has or has not odd permutations: $h_{n-1}h_1 \wr \mathrm{Grf}(G) = \mathrm{Grf}(G)$;

$$h_{(n!/\mu)-1}\, h_1 \wr [\mathrm{Grf}(H) \wr \mathrm{Grf}(G)] = \mathrm{Grf}(H) \wr \mathrm{Grf}(G).$$

7. *Determination of the G. R. F. from Given Generators.* If two symbols α and β occur in the same cycle of one of the given generators of a group G whose G. R. F. is to be determined, they belong to the same transitive constituent of G. By examination of the given generators from this point of view, the number θ of transitive constituents can be found. Writing $\mathrm{Grf}(G)$ in the general form $(1/m)\sum[N(G; p_1^{\pi_1}p_2^{\pi_2}\cdots)s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots]$, the relation $h_{n-1}h_1 \bigotimes (1/m)\sum[N(G; p_1^{\pi_1}p_2^{\pi_2}\cdots)s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots] = \theta$ gives an equation with m and the N's as unknowns. Again, taking any known group H of degree n and order μ, we can form a function ϕ, of the original n symbols, which is invariant under H, and then determine the forms of the new generators, each containing now $n!/\mu$ symbols, and corresponding to the original generators, which generate the isomorph whose G. R. F. is $\mathrm{Grf}(H) \,\delta\, \mathrm{Grf}(G)$; and another equation is given by

$$h_{(n!/\mu)-1}\, h_1 \bigotimes [\mathrm{Grf}(H) \,\delta\, \mathrm{Grf}(G)] = \mathrm{Grf}(H) \bigotimes \mathrm{Grf}(G) = \theta',$$

θ' being found by inspection of the new generators just as θ was found previously. Continuing in the same way, using various other groups for H, a sufficient number of equations can in theory be found to determine m and all the N's. In practice some of the isomorphs will be of inconveniently high degree, a difficulty which can be mitigated to some extent by choosing the groups H of as high order as possible.

It is not necessary, however, to set up as many equations as there are unknowns. For without actually evaluating $\mathrm{Grf}(H) \bigotimes \mathrm{Grf}(G)$, we know that it is a positive integer. Also $s_{\lambda_1}^{l_1}s_{\lambda_2}^{l_2}\cdots \bigotimes \mathrm{Grf}(G)$ is a positive integer or 0 for every partition $\lambda_1^{l_1}\lambda_2^{l_2}\cdots$ of n; while if H' is a sub-group of H, it is easily shown that $\mathrm{Grf}(H') \bigotimes \mathrm{Grf}(G) \geqq \mathrm{Grf}(H) \bigotimes \mathrm{Grf}(G)$. These and similar relations furnish a variety of congruences and diophantine inequalities which, in connection with a partial set of definite equations, will frequently suffice for the determination of $\mathrm{Grf}(G)$.

Even when so simplified, it may be doubted whether the method offers any substantial advantage over the alternative, frequently impracticable of course, of actually deriving the whole set of operations of the group from the given generators and then forming the G. R. F. by counting the operations of various cycle-partitions. It is here given rather as illustrating a principle on which it may be possible to base an effective method. For the present we must in practice rely on special devices when dealing with groups of high orders.

8. *Possible Algebraic Forms of G. R. F.'s.* The congruences and diophantine inequalities referred to in § 7 constitute a set of *necessary* conditions that a given symmetric function be the G. R. F. of an existent group. By means of them we can determine the possible forms of G. R. F.'s having specified properties, as for instance those belonging to groups of given degree and order. We can often thus demonstrate the non-existence of certain classes of groups, or narrow the possibilities to one or a few forms. We cannot however so demonstrate the existence of any proposed group, and a completely *sufficient* set of purely algebraic conditions has not been discovered, though nothing indicates that this would be impossible. Such methods might be useful in checking for omissions the lists of groups compiled by Cayley and others (*Quarterly Journal of Mathematics,* Vol. XXV, p. 71, etc.).

WAYNE, PA.
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Closure of the Tangential Process on the Rational Plane Cubic.

By F. E. Allen.

1. In a series of recent papers, Winger * has developed various features of the satellite theory of the plane cubic by use of the parameter form of representation and involutions on rational curves. Classic theorems and extensions of satellite properties are proved with great ease and beauty.

A prominent feature of these papers is the discussion of systems of contact curves. Of special interest is Winger's conic N, the envelope of lines joining pairs of contact of tangents from points of the rational cubic, and tangent to the cubic at the sextactic points. This conic is one of a pencil of conics which touch the nodal tangents where they meet the line of inflexions. Since both the cubic and each conic of this pencil are invariant under a dihedral G_3, the pencil and the conic N play an important part in the construction of special and general sets of the group.

The problem of closure of the tangential process on an elliptic cubic or higher curve has been extensively discussed from the number-theory standpoint,† leading to the determination of sets of real, rational points on a curve. With this has been developed the configurations of polygons, both inscribed and circumscribed to a curve of third order or higher.

The present paper takes up the problem of determining those points on a rational cubic which coincide with their nth tangentials from the more general projective standpoint. It is found that the pencil of conics above described is of fundamental importance.

The problem exists for nodal cubics only since the nth tangential of a point on a cuspidal cubic approaches the cusp as a limit.‡

2. To determine those points on a rational cubic which coincide with their third tangentials.

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* *Bulletin of the American Mathematical Society*, October, 1918, November, 1919; *American Journal of Mathematics*, Vol. 42 (1920).

† Sylvester, *American Journal of Mathematics*, Vol. 2, 3; Story, *Ibid.*, Vol. 3; Picquet, *Journal de l'École Polytechnique*, Vol. 54; Porter, *Transactions of the American Mathematical Society*, Vol. 2.

‡ Salmon, *Higher Plane Curves* (2nd ed.), p. 177.

The equation used is that of the crunodal cubic in the form

$$x^3 + y^3 - 3xyz = 0,$$

where the nodal tangents are the x and y axes respectively, and the line of inflexions is the z-axis. The interchange of real and imaginary elements for the acnodal case may be made at any time.

The parameter form is

$$x : y : z = 3t^2 : 3t : t^3 + 1.$$

The first tangential of the point t is $-1/t^2$,† the second tangential is $-t^4$, etc., the nth tangential being $-t^{(-2)^n}$.

(a) If $t = -1/t^2$, $\quad t^3 + 1 = 0.$

The roots are the parameters of the inflexions.

(b) If $t = -t^4$, $\quad t(t^3 + 1) = 0.$

The roots are the parameters of the node and of the inflexions.

(c) If $t = -1/t^8$, $\quad t^9 + 1 = 0.$

$$(t^3 + 1)(t^6 - t^3 + 1) = 0.$$

After removing the factor which corresponds to the inflexions, the six remaining points are seen to lie on a conic since they belong to the involution $s_6 = 1$. These are the six "coincidence points" of the rational cubic.‡ This is easily identified with the conic

$$xy - 3z^2 = 0,$$

which belongs to the pencil already described. It is a special case for the nodal cubic of the conics described by Porter † in his discussion of the coincidence points of the general cubic.

The following construction may be shown for this coincidence-conic: It passes through the points in which a line joining the node to an inflexion meets the tangents at the sextactic points of the other two inflexions.

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* Winger, *loc. cit.*

† Halphen, *Mathematische Annalen*, Vol. 15.

‡ *Loc. cit.*

3. To determine the satellite conic of a conic of the system

$$xy - \lambda z^2 = 0.$$

If the satellite belongs to the same system, its equation is of the form

$$xy - \lambda' z^2 = 0.$$

If the primary point is $(3t^2, 3t, t^3 + 1)$ the tangential is $(3t^2, -3t^4, t^6 - 1)$. The t-equations giving the parameters of the points of intersection of these conics with the cubic are

$$t^6 + [2 - (9/\lambda)]t^3 + 1 = 0. \tag{1}$$

$$t^{12} - [2 - (9/\lambda')]t^6 + 1 = 0. \tag{2}$$

Equation (2) must contain (1) as a factor, the other factor being the function corresponding to its other primary, found by replacing t by $-t$. Eliminating t, there results

$$\lambda' = \lambda^2/(4\lambda - 9),$$

which transforms the parameter of a conic of the system into that of its satellite.

$$\text{If } \lambda' = \lambda, \ \lambda = 0 \text{ or } 3.$$

$\lambda = 0$ gives the degenerate case of the nodal tangents. Hence, the only proper conic of the pencil which is its own satellite is the coincidence-conic which cuts out of the cubic two cyclic groups of points which coincide with their third tangentials:

$$t = e^{\pi i/9}, \quad e^{7\pi i/9}, \quad e^{13\pi i/9},$$
$$t = e^{5\pi i/9}, \quad e^{17\pi i/9}, \quad e^{11\pi i/9}.$$

To $\lambda = 9/4$, Winger's conic N, corresponds $\lambda' = \infty$, the line of inflexions doubled.

4. To determine the points which coincide with their fourth tangentials.

$$t = -t^{16} \quad t(t^3 + 1)(t^{12} + \cdots + 1) = 0.$$

The solution lies in the two conics of the third factor. This will factor into

$$t^6 + [2 - (9/\lambda)]t^3 + 1 = 0 \tag{1}$$

with roots, t_1, $1/t_1$, ωt_1, ω/t_1, $\omega^2 t_1$, ω^2/t_1, and

(2) $$t^6 + [2 - (9/\lambda')]t^3 + 1 = 0$$

with roots t_2, $1/t_2$, ωt_2, ω/t_2, $\omega^2 t_2$, ω^2/t^1, and

Let $$t = e^{(2k+1)\pi i/15} \qquad 2k+1 \not\equiv 0 \pmod 5$$

Substituting in (1) and equating to zero the real (or imaginary) part:

$$\cos 2(2k+1)(\pi/5) + [2 - (9/\lambda)] \cos (2k+1)(\pi/5) + 1 = 0,$$
$$2\cos^2(2k+1)(\pi/5) + [2 - (9/\lambda)] \cos (2k+1)(\pi/5) = 0.$$

$$\lambda = \frac{9}{4\cos^2(2k+1)(\pi/10)}$$

$$\lambda_1 = \frac{9}{4\cos^2 18^\circ} \qquad \lambda_2 = \frac{9}{4\cos^2 54^\circ}.$$

By the formula

$$\lambda' = \frac{\lambda^2}{4\lambda - 9},$$

it may be shown that λ_1 is the satellite of λ_2, and conversely. The cyclic groups of points are shown by the following table:

$$t = e^{(2k+1)\pi i/15}$$

λ	k	t	k	t	k	t
λ_1	1	$e^{3\pi i/15}$	3	$e^{7\pi i/15}$	8	$e^{17\pi i/15}$
λ_2	4	$e^{9\pi i/15}$	0	$e^{\pi i/15}$	5	$e^{11\pi i/15}$
λ_1	13	$e^{27\pi i/15}$	6	$e^{13\pi i/15}$	11	$e^{23\pi i/15}$
λ_2	10	$e^{21\pi i/15}$	9	$e^{19\pi i/15}$	14	$e^{29\pi i/15}$

5. Points which coincide with their fifth tangentials.

$$t = -1/t^{32} \qquad t^{33} + 1 = 0 \qquad (t^3+1)(t^{30} + \cdots + 1) = 0.$$

The last factor yields five conics by the process of the preceding section.

$$t = e^{(2k+1)\pi i/33} \qquad (2k+1) \not\equiv 0 \pmod{11}$$

$$\lambda = \frac{9}{4\cos^2(2k+1)(\pi/22)}$$

$$\lambda_1 = \frac{9}{4\cos^2(\pi/22)}, \quad \lambda_2 = \frac{9}{4\cos^2(9\pi/22)}, \quad \lambda_3 = \frac{9}{4\cos^2(7\pi/22)}$$

$$\lambda_4 = \frac{9}{4\cos^2(3\pi/22)}, \quad \lambda_5 = \frac{9}{4\cos^2(5\pi/22)}.$$

λ	k	k	k	k	k	k
λ_1	0	11	22	10	21	32
λ_2	15	26	4	28	6	17
λ_3	18	29	7	25	3	14
λ_4	12	23	1	31	9	20
λ_5	24	2	13	19	30	8

6. Points which coincide with their sixth tangentials.

$$t = -t^{64} \qquad t(t^{63}+1) = 0 \qquad t(t^{3}+1)(t^{60}+\cdots+1) = 0$$
$$t = e^{(2k+1)\pi i/63} \qquad 2k+1 \not\equiv 0 \pmod{21}$$

If $2k+1 \equiv 0 \pmod 7$ the solutions are those of a previous case, $t^9+1=0$. Hence there are 9 conics for the remaining 54 values of k.

$$\lambda = \frac{9}{4\cos^2(2k+1)(\pi/42)} .$$

There are 6 conics which form a cyclic satellite-group, and 3 conics such that each coincides with its third satellite conic, as follows:

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
$k=$	0	30	33	27	39	15
$k=$	20	53	50	56	44	5
$k=$	21	51	54	48	60	36
$k=$	41	11	8	14	2	26
$k=$	42	9	12	6	18	57
$k=$	62	32	29	35	23	47

	λ_7	λ_8	λ_9	λ_7	λ_8	λ_9
$k=$	4	22	49	58	40	13
$k=$	25	43	7	16	61	34
$k=$	46	1	28	37	19	55

The following values of k determine points on the coincidence conic, $\lambda=3$:

3, 24, 45; 17, 59, 38.

7. General Case. Points which coincide with their nth tangentials.

$$t = -t^{(-2)^n}$$

Case 1. n even: $t = -t^{2^n}$

$$t(t^{2^n-1}+1) = 0$$
$$2^n - 1 \equiv 0 \pmod 3$$
$$t(t^3+1)(t^{2^n-4}+\cdots+1) = 0$$
$$2^n - 4 \equiv 0 \pmod{12}.$$

The number of conics is

$$(2^n-4)/6 = (2/3)(2^{n-2}-1)$$

including as special cases all solutions for the prime factors of n.

Case 2. n odd. $t = -(1/t^{2^n})$ $t^{2^n+1}+1 = 0$

$$2^n + 1 \equiv 0 \pmod 3$$
$$(t^3+1)(t^{2^n-2}+\cdots+1) = 0$$
$$2^n - 2 \equiv 0 \pmod 6.$$

The number of conics is

$$(2^{n-1}-1)/3$$

which includes the solutions for the prime factors of n.

If n is an odd prime, this formula gives the exact number of proper solutions. By Fermat's theorem

$$2^{n-1} - 1 \equiv 0 \pmod n$$

and hence the number of cyclic groups of n conics is

$$(2^{n-1}-1)/3n \qquad n \neq 3.$$